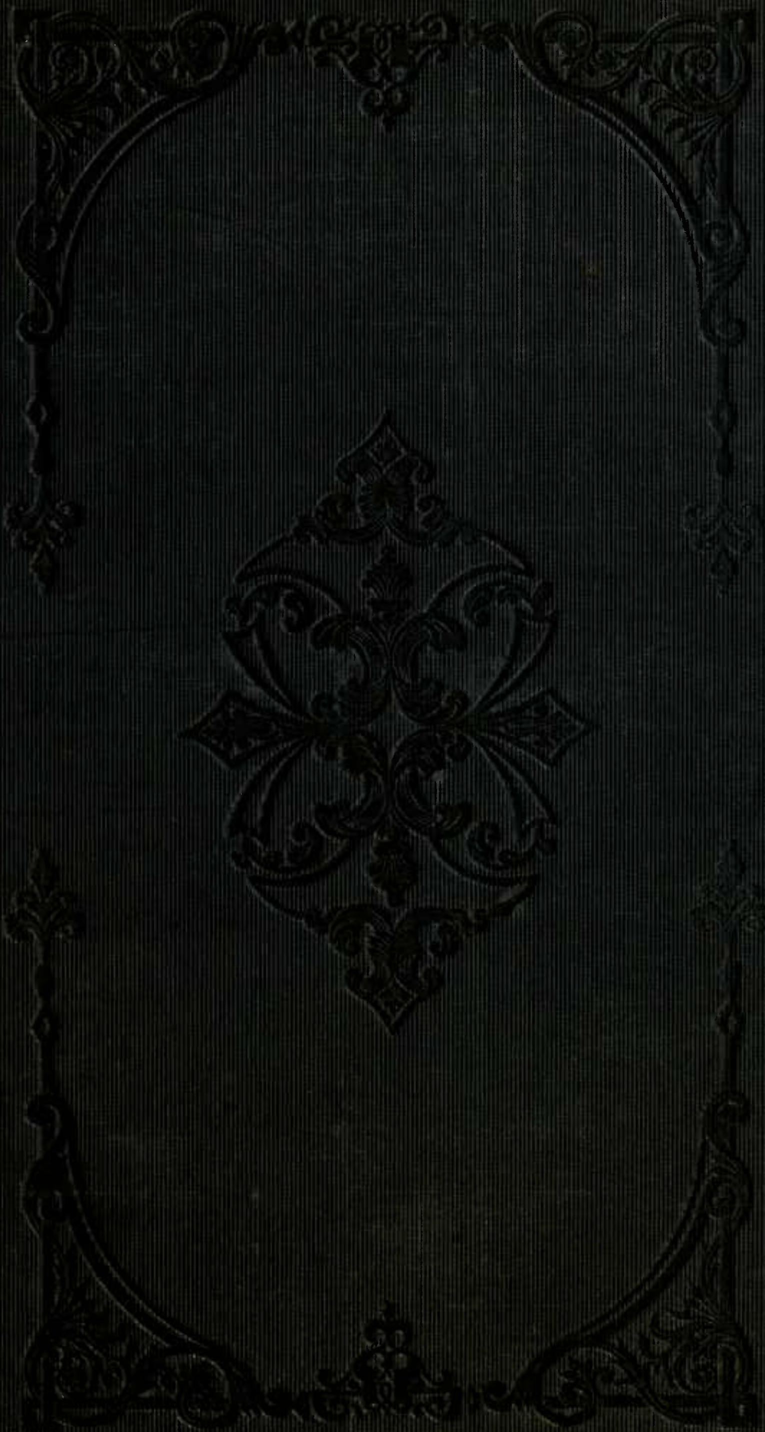

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AN
ELEMENTARY TREATISE
ON
PHYSICAL ASTRONOMY.

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BY

ROBERT WOODHOUSE, A.M. F.R.S.

FELLOW OF GONVILLE AND CAIUS COLLEGE.

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PREFACE.

It must be in compliance with custom, rather than from any distinct view of good likely to result, when an Author begins his Work by defining the Science he means to treat of. A definition is not easily laid down. It is not difficult, indeed, to define a branch of science in general terms; but such are seldom intelligible to the Student. If we enumerate what is too summarily expressed, and explain a general statement by detailing certain cases comprehended under it, we, probably, forestall what belongs to the body of the Work. We attempt to do immaturely what, it is almost certain, will be done imperfectly; and this without an adequate advantage; for, a definition such as we allude to, entailing no consequences, is not required in the beginning of a Work: at the end it is unnecessary.

But if a Student does not require, as essential to the perusal of a Work, a formal definition of its drift and nature, an Author will gladly be absolved from giving one. He cannot but wish to avoid such slippery ground. For, should he restrict himself, as it is usual, to few terms, he is in danger of defining too largely, or too partially, or too vaguely. If it be said, the object of Physical Astronomy is the explanation of heavenly phenomena, the definition is too wide: if merely of the laws of the motions of the Stars, too restricted: if of those laws on mechanical principles, too vague and indistinct: if of their causes, too presumptuous and illusive.

Even Newton's Theory, perfect and excellent as it is, and on which Physical Astronomy is founded, does not pretend to explain the *causes* of the phenomena of the heavenly bodies. It rather explains why they may be reduced to the same class; which is an object more simple and distinct. The two points on which the theory rests, are, first, that every particle of matter attracts, and, consequently, that two particles mutually attract each other; the second point is, that, if the distance between the particles vary, the *attraction* will vary proportionally to the inverse square of the distance. The first of these is called the *Principle*, the second the *Law of Gravity*.

But the terms *Attraction* and *Gravity*, although they seem borrowed from the language of Causation, are not meant to signify any agency or mode of operation. They stand rather for a certain class of like effects, and are convenient modes of designating them. One of these effects is the space fallen through by a heavy body at the Earth's surface: another is the *deflection* of the Moon from the tangent of her orbit towards the Earth; and, in every case, *gravity*, or *attraction* is expounded by a like space or deflection. If, on analysing a phenomenon of a revolving planet, we can detect such space or deflection taking place towards the attracting body, we have found out all that is meant by attraction. If, for instance, we can so resolve an arc of the Moon's orbit into the elements producing it, that two of them being the Moon's velocity and direction, the other two shall be spaces or deflections towards the Earth and Sun respectively: the former distance proportional to the Earth's mass and the inverse square of the distance of the Earth and Moon, the second proportional to the Sun's mass and the inverse square of the distance of the Sun and Moon, we have found out all that is necessary to be understood by the Earth's *attracting* the Moon, and the Sun's *attracting* the Moon: or, in other words, by the Moon's *gravitating* to the Earth, and *gravitating* to the Sun: although the latter part of this expression, so applied, is contrary to the technical and conventional language, which, for the sake of distinction, it is found convenient to employ.

It is thus, by resolving a phenomenon, that we may form a notion of gravity and attraction: and we may obtain an equally distinct notion by the reverse process. Draw, for instance, from the Moon towards the Earth and Sun, two lines, representing, respectively, according to the prescribed conditions, (see p. 10.) the attractions of the Earth and Sun: then combining these with the lines representing the Moon's velocity, &c. according to the principles of Dynamics (those principles by which we estimate a body's motion in a parabolic curve, and the oscillation of a pendulum) the result will be the described arc of the Moon's orbit.

According to Newton's Theory, like results take place throughout the planetary system: each planet is attracted by all the rest, and their attractions are to be expounded similarly to the attractions that have just been spoken of.

This is a general statement which is easily made; but the actual finding of the results, must, it is plain, be a most difficult research. The attracted and attracting bodies, both with regard to their relative situation and the intensities of their mutual attractions, are in a state of perpetual change. Their *Configuration*, as it technically is called, is for ever varying: and whether we investigate the arc of an orbit, or a change in its dimensions, the result must be the modified effect of many forces, that, during unequal times and with varying intensities and directions, have been sometimes conspiring, and at other times counteracting each other.

This may give us some idea of the difficulties of Physical Astronomy, they are indeed so great, that, if met in their full extent, they cannot be completely overcome.

But there are various means for lessening and avoiding them: some devised, others naturally presenting themselves. The Moon's motion, for instance, we have considered to depend on her velocity and direction, and on the forces of the Sun and Earth. But, according to Newton's Principle of Universal Gravitation, every planet in the system must, like the Sun, attract the Moon. Each planet, however, attracts so much less forcibly, that their combined effect may be neglected and the research of the Moon's motion facilitated by being freed from their computation. As the merely *analytical* investigation of the inequalities caused by the smallest body of the system, is the same as that of those caused by the largest, it is a point of no very easy consideration to know what ought to be retained for computation, and what may be cast aside. There is no process so complete, when the object of research is the motion of the Moon or of a planet, in which all conditions are retained and made account of: some at least are rejected or made more simple: not solely from the will of the mathematician, or for mere convenience; but from the absolute insufficiency of the art of Calculation.

We may at this point discern the several sorts of explanation that may be admitted into a Treatise on Physical Astronomy. For, as in the case of the Moon, the attractions both of the Sun and planets are much less than the attraction of the Earth round which, as a central body, the Moon revolves, so in every other case, the attraction of the *external* on the revolving body, is very much less than the attraction of the central body. The orbit, therefore, and the laws of motion of the revolving body must be nearly the same as if the external body or bodies were abstracted. We may, therefore, as a first step, and as an approximate result, investigate the orbit in this latter case: or, we may make a step in advance towards a better explanation, by considering the actions of the external bodies to be extremely small: or, by taking account of some and by neglecting others: or, for the sake of facility, by changing and simplifying the conditions under which they act: so that great variety of explanations (as far as difference of degree constitutes variety) present themselves to an Author for his adoption.

The first in this series of explanations might be easily obtained, but would be most imperfect. If we so curtail Newton's Theory of Gravitation as to abstract, from Mars revolving round the Sun, the attractions of the other planets, we may, indeed, still establish by it the greatest of Kepler's discoveries, (the elliptical orbit of Mars) and his Laws respecting Areas and Periods.

But other phenomena must, on such partial principles, remain unexplained: not solely those which have been detected by the nicety of modern research, but such as the principal Lunar inequalities, which, previously to Newton, had been discovered by Tycho Brahé and others.

Of explanations towards the other part of the series there is no termination. Whatever be the labour of research, we can never arrive at a complete explanation of the motions of the heavenly bodies; the reason is, their *natural* complication, or (for we may use either statement) the imperfection of the art of calculation. The solutions in Physical Astronomy are approximate ones, but, since approximate, capable of being conducted, by continuation of process, to any point short of absolute exactness. We may, instead of the twenty-five equations which of late times have been added to the five by which Tycho Brahé determined the Moon's place, deduce five hundred; and compute coefficients so minute, that observations made for centuries, with instruments more perfect than what are now used, will not be able to verify.

Computations so conducted would contain a great deal of useless accuracy and would add nothing to speculative truth. Yet no rules can be laid down for limiting them. In order to know how far it is useful to extend our calculations, we must refer to observation. If its errors should exceed one or two seconds, it would be useless to compute coefficients that cannot exceed the tenth of a second.

But the question is as doubtful as the former one, if it be enquired at what gradation an Author ought to stop between inordinate calculation, and a too artificial and assumed simplicity of explanation. It may be said, either the state of Science, or the wants of the Student ought to determine him; but the rule is altogether ambiguous. And even if we suppose a Work planned and begun under such considerations, it will not be so continued and terminated. When it is once begun, we cease (such is the fact) to conduct it by looking at its drift and scope.

Of the present Work it would be a fruitless attempt to describe its precise character: it is very far from being, according to the common acceptation of terms, elementary. On the other hand, its processes, being in many instances, stopped short of that exactness which they admit of, will not immediately serve the construction of Lunar and Planetary Tables.

If simplicity of demonstration consist in the ease with which we are enabled to pass over its steps without regard to their number, then, perhaps, it is the Author's fault if all demonstrations are not equally simple. There should then be no difference, in that respect, between Kepler's Law of the Equable Description of Areas, the *Acceleration* of Areas, and the *Progression* of the Lunar Apogee. But if, as it is commonly thought, we depart from the simplicity of proof, by increasing the number of its steps, then, whether a demonstration can be simple or not, must depend on the nature of the thing to be demonstrated. Under this point of view very few of the demonstrations in Physical Astronomy can be simple. Its objects of research are abstruse. They do not often lie near the surface: and, what is recondite we must be at the labour of searching for. We cannot go from the principle to the result by a shorter route than the direct series of consecutive steps. Much less in Physical Astronomy, than in pure Geometry, is there any privileged access to Science.

Physical Astronomy, then, is in this predicament. Its solutions can neither be simple nor complete. They are prevented from being simple by the intricacy of the object of research, and from being complete by the imperfection of the Art of calculation.

The solutions in Physical Astronomy are now longer than they were at the time of its rise: and, therefore, under a certain point of view, it would seem as if there were wanting, with regard to them, that usual effect of time by which conciseness is conferred on scientific processes. But, the fact is, although the research be now more intricate, the things sought for, the orbit of a planet, and the laws of its motions, are more accurately traced out than they were heretofore.

It became necessary to trace them out more accurately by an improved analysis, when they were more minutely noted as phenomena by improved observations. The necessity created the means.

A refined art of observing phenomena would not suit with a rude science of computing them from theory; they would then admit of no strict explanation; for that must consist in the agreement of the results of theory and observation. But an explanation and agreement might seem to take place, and at several stages, whilst theory and observation progressed, at nearly an equal rate, towards improvement. For instance, when by the examination of observations, the orbit of Mars was judged to be elliptical, the phenomenon seemed to be explained by Theory, when the latter demonstrated the necessary description of an ellipse by a projected body attracted to a centre by a force varying inversely as the square of the distance. But in this case, neither theory nor observation were strictly correct. The phenomenon was inaccurately noted, and its explanation was given on partial principles. The orbit of Mars is not strictly elliptical, nor is the case of a revolving body attracted to a centre by a force varying inversely as the square of the distance, exactly exemplified by Mars revolving round the Sun. The orbit, however, is nearly elliptical, and the Sun's is the paramount attraction acting on Mars. But the slight deviations from an elliptical orbit, detected by nicer observations, would require, for their explanation, a new or a modified theory. The former theory excluded deviations altogether. The sole source of attraction being supposed to be the central body, the orbit of the revolving body, is not approximately, but exactly, an ellipse: but other heavenly bodies, besides the Sun, attract Mars. The former theory, therefore, was partial because it excluded their agency. In order, however, to produce an agreement, at this second stage, between theory and observation, it becomes necessary to compute the deviations or inequalities caused by *planetary attraction*. This is a far more difficult problem than that of the description of an ellipse, and requires a more refined art of calculation.

This therefore is the criterion of a true theory, and by which Newton's will stand : that, by legitimate inferences from its principles, it should constantly supply those new demands of explanation which the improved observation of phenomena, that are its objects, renders necessary.

But although the theory itself may remain as simple as ever, its processes become thus, by the advancement of science, more and more intricate. The demonstration of Kepler's Laws on the restricted condition of the sole agency of a *centripetal* force, is more simple than that of their derangement according to the real circumstances that take place in nature. If we feign the Sun and Jupiter to be the only two attracting bodies, it is easy to shew that the latter must perpetually revolve round the former in a perfectly elliptical orbit. But if we restore the system to its truer state, and introduce Saturn, then it is very difficult to shew that this latter planet will so cause Jupiter to deviate from the elliptical orbit, as, after a certain period, again, to enter it at the point of previous deviation, and to reiterate his irregular course.

The more curious the phenomena the more intricate their theoretical investigation : and, as the fact is, the art of observing phenomena, and the science of computing them, have nearly kept pace with each other : and, were this the occasion, we might hence derive some arguments for the propriety of arranging and expounding the resources of Physical Astronomy according to the historical order of their production and accumulation.

But such an history, could it be obtained, would be chiefly one of Newton's mind : of its successive discoveries and inventions : for, what is unexampled in the annals of Science, he not only founded, but without help established, almost completely, his System of Gravitation.

He made in Physical Astronomy what an acute writer * considers the test of a rare excellence, not only the first but great steps.

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* 'Et que ne coutent point les premiers pas en tout genre ! le merite de les faire dispense de celui d'en faire de grands.' Dalembert, *Discours Prelim.*

It cannot, therefore, be asserted that Newton ever supposed, even in his first conjectures, the orbit of Mars to be perfectly elliptical, or that the equable description of areas would exactly take place. These might be, indeed, results which his mind at first acquiesced in: or, he might consider them merely as first steps in his theory; as the most simple and intelligible of the arguments by which the Principle and Law of Gravitation were to be established. It was natural he would begin with such. Yet simple as they may now appear to be, the proofs, on mechanical principles, of the equable description of areas, of the elliptical orbit of a planet, of the squares of the periods of planets varying as the cubes of their mean distances, must have appeared wonderful discoveries to Newton's contemporaries. Less simple results of his theory, would not, perhaps, have excited an equal interest and attention.

But Newton passed on to farther results. His Theory of Universal Gravitation would not suffer him to stop at the elliptical form of a planet's orbit. If Mars's orbit were an ellipse, the Sun's being the sole attracting force, the Moon's orbit, by the same proof, would be an ellipse, were the Moon attracted solely by the Earth. But the Sun's force so paramount in the case of Mars, could not but act on the Earth and Moon, and might act on them unequally; and if so, the inequality of attraction towards the Sun, would become, in investigating the Moon's orbit, a condition additional to that of the Earth's centripetal force. The orbit and the laws of motion could not be the same as if that condition were abstracted: they, therefore, could not be elliptical: and this theoretical inference, was, in some degree, confirmed by observation: for the Moon's *Variation*, *Evection*, and *Annual Equation*, are merely so many terms for deviations from elliptical motion. Whether these *Inequalities* could be exactly accounted for from the above inequality of attraction (which technically is the Sun's disturbing force) was a matter of calculation.

On the agreement then of the results of such calculation with the phenomena which Tycho Brahé and others had detected in the Moon's motion, the second set of arguments for the truth of Newton's System would depend. The Lunar inequalities were to be used in proving what Mars's elliptical orbit had, in part, proved; but the proofs would be of a higher scale, and founded, by more refined processes of calculation, on less partial principles: for, on the exact principles by which Mars's orbit is shewn to be an ellipse, the Moon's apogee must be quiescent.

The series of proofs, of which we have spoken, deserves a more minute consideration. Their order and combination constitutes the system of Physical Astronomy.

That system, as it has been already mentioned, has no remote origin: it is to be dated from Newton. The writings of former philosophers neither describe nor suggest such a system. We may find, indeed, here and there, a scattered hint of some such principle as Gravity: but nothing (which is the main point) said of its universality and law. In this respect it had escaped even the glances of random conjecture. No theory ever had fewer anticipations than Newton's. It was, to use a phrase of my Lord Bacon, completely *ex partu ingenii*.

But although the theory was novel, yet science, at the time of its invention, abounded with results and methods. The celestial phenomena had previously been diligently and accurately observed. Plane Astronomy was rich not merely in registered elongations, right ascensions, declinations, &c. but in several curious results deduced from them: such, for instance, were the equable description of areas, the elliptical orbits of planets, the progressions of the aphelia, the regression of the nodes of the Moon's orbit, the change of the inclination of its plane, and, above all, the Lunar inequalities of the *Variation*, *Evection* and *Annual Equation*, with the laws of their increase and decrease. These phenomena (for so they may be called, although they are not the immediate objects of observation) had been discovered by Ptolemy, Tycho Brahé and Kepler, but not classed together as like effects, nor explicated on mechanical principles.

Other phenomena, however, not astronomical, had, previously to Newton, been so explained. Galileo, on the hypothesis of a constant source of acceleration, had deduced the laws of falling bodies, and (what had a closer connexion with Newton's pursuits) had explained the mode of describing a parabola by compounding the projectile force with that of gravity. Huygens had gone farther. He had invented theorems relating to the centripetal forces of bodies describing circles, and to the involutes of curves. These led the way to Newton's first researches. There was only one step between them and that theorem which assigns the law of force whatever be the curve described.

The explanations of the laws of falling bodies, and of the descriptions of a parabola and circle by two forces, were, taking an extended signification of the term, explanations on *mechanical principles*. On like principles, Newton proceeded to explain the planetary phenomena. Of these one of the most simple and remarkable is a planet's elliptical orbit. Such an orbit might be conceived to be described as Galileo and Huygens had conceived a parabola to be. But, in order to go beyond the mere conception of the mode of description, it was necessary to possess a theorem for determining the law of force tending to a point or centre within the ellipse. No such theorem existed: but Newton invented one by a most dextrous combination of those two that related to the circle of *curvature* and the centripetal force of a body revolving in a circle.

The theorem, however, which Newton invented, on this occasion, was not restricted to an ellipse, but applied generally to any curve, whatever were the law of its description. From its application to an ellipse, it resulted that the law of force, tending to its focus, was inversely as the square of the distance: and it easily followed, by a converse process, that a body projected obliquely to a line joining it and the centre of force (the force varying inversely as the square of the distance) would describe an ellipse round that centre.

This may be considered as the first instance of that law which is frequently called, for distinction, the *Law of Nature.* It was, in the above instance, the means of shewing that to be a necessary truth which Kepler, by observation, had ascertained to be a fact.

The fact, however, appeared to Kepler so curious and important that he viewed it as a Law of Nature. A second law he detected in the equable description of areas; and that Newton shewed to be a necessary consequence of the mode by which a curve was conceived to be the result or combination of two motions: one uniform and in the line of the body's motion, the other constantly directed towards the centre round which the areas are described, and being always, at the same distance from the centre, of the same magnitude. The demonstrations of these two laws are contained in the twelfth Proposition of the third Section, and first Proposition of the second. Kepler had observed the equable description of areas to take place in the apsides of a planet's orbit. Newton's proof was independent of the body's place, and, also, of the law of the force: it required, however, as an essential condition, that the force should be centripetal.

By combining the Propositions on which the proofs of the two preceding Laws of Kepler depended, Newton obtained a third result, which is, the squares of the periods of bodies describing ellipses about a centre of force situated in the focus are proportional to the cubes of the greater axes of those ellipses. And this constitutes Kepler's third Law.

The three Laws of Kepler were thus explained by Newton on mechanical principles. Their demonstrations were, in the first instance, communicated to the Royal Society at the request of Halley; and afterwards published in the *Principia* *.

We must carry ourselves back to the time of Newton and consult contemporary writings in order to appreciate rightly the merit of these great discoveries. But, important as they are, they contain nothing relative to the doctrine of Universal Gravitation, which is the main basis of Newton's fame. Indeed the Propositions by which Kepler's Laws are demonstrated, together with others of the second and third Sections, may be viewed as so many merely mathematical Propositions, and (if we except one passage *) not applied, in the place where they are inserted, to the System of the Universe.

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* See the first, eleventh and fifteenth Propositions of the first Book.

But these Propositions were inserted in the second and third Sections to be afterwards referred to †. Now it follows from the thirteenth Proposition, that the quantities of centripetal force acting on Mars in different points of his orbit, are to each other inversely as the squares of the distances of those points from the Sun. By a like inference, the forces on Jupiter, his orbit also being supposed to be elliptical, similarly vary. But it does not immediately follow from these results, that the centripetal forces urging Mars and Jupiter, are to each other inversely as the squares of their respective distances from the Sun. The absolute quantities of those forces (estimating them by their effects at the same distance from the Sun), might, for any thing contained in the Proposition referred to, be different. An intermediate step, therefore, is necessary, in order to shew that Mars and Jupiter, if placed at equal distances from the Sun, would, in the same time, fall through equal spaces; and such an one is found in that law which exists between the periods and mean distances. If the square of Jupiter's periodic time be to the square of Mars's, as the cube of Jupiter's mean distance from the Sun to the cube

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[*] 'Casus Corollarii sexti obtinet in corporibus cœlestibus (ut seorsum collegerunt etiam nostrates Wrennus, Hookius et Hallæius) et propterea quæ spectant ad vim decrescentem in duplicatâ ratione distantiarum a centris, decrevi fusius in sequentibus exponere.'

[†] In libris præcedentibus principia philosophica tradidi, non tamen *philosophica* sed *mathematica tantum,* ex quibus videlicet in rebus philosophicis disputari possit, &c. Eadem tamen ne sterilia videantur, illustravi scholiis quibusdam philosophicis, &c. *Newtonus de Mundi Systemate.*

of Mars's, then at equal distances from the Sun they would be urged towards him by equal forces : for such equality of force is an essential condition of Newton's fifteenth Proposition of the third Section.

Now if, at equal distances from the Sun, Mars, Jupiter, Venus, &c. were urged towards him by equal forces, or if they then *gravitated* equally to him, it was no very improbable supposition that these forces arose from some attraction in the Sun. It, at the least, involved no contradiction, to suppose that the matter or particles of the Sun *caused* the gravitation of the planets towards him. The mode by which this was effected formed no part of such supposition, nor was, in any sort, implied by it.

But the supposition that the matter of the Sun caused the gravitation of the planets, if it involved no contradiction, would naturally give rise to conjecture and enquiry. If the Earth and Jupiter gravitated to the Sun by virtue of his attracting particles, would not Jupiter's satellites towards Jupiter, and the Moon towards the Earth, *gravitate* from like particles resident in Jupiter and the Earth ? and might not these gravitations, at equal distances, be in proportion, respectively, to the number of particles, or the masses of the central bodies ? At this point of Newton's research, if we were permitted to feign its theoretical history, might be supposed to have arisen the momentous question concerning Universal Gravitation : a question of great extent, and not admitting of any summary determination.

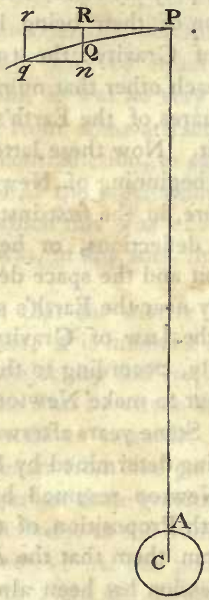
Not summarily to be decided on, except its connected theory were false : in that case one *impugning* instance would overthrow the theory : but if true, a thousand instances would only *tend* to establish it. The proof of the truth of Newton's Theory is only the accumulation of individual arguments, derived from various instances, and all conspiring : and the first in the series of arguments, to prove that all bodies gravitated, the one towards the other, was derived from the *Moon's Gravitation.*

The drift of this first argument was to shew that the descent of a heavy body near the Earth's surface, and the *deflection* of the Moon from the tangent of her orbit, were like effects, or of the same class; or, which would make the analogy closer, that the latter deflection, and the deflection from the tangent of a parabola described by a heavy body projected near the Earth's surface were like effects. The criterion of their being like effects consists in their obeying the Law of Gravity: the two deflections, therefore, ought to bear to each other that numerical relation which subsists between the squares of the Earth's radius, and of the radius of the Moon's orbit. Now these latter quantities were inaccurately known at the beginning of Newton's researches: our great philosopher, therefore, in the first instance, found that the relation between the two deflections, or between the sagitta of an arc of the Moon's orbit and the space described (in the same time as the arc) by a body near the Earth's surface, was not such as it ought to be, were the Law of Gravity true. The relation was nearly, but not exactly, according to that law. But the difference was quite sufficient to make Newton suspend his decision on the truth of the law. Some years afterwards, however, the dimensions of the Earth being determined by Picart more accurately than they were before, Newton resumed his investigation, (such as we find it in the fourth Proposition of the third Book of the *Principia*), and found from them that the *Moon gravitated.* The signification of that expression has been already explained: if it required farther illustration, we might say, that, a heavy body removed to the Moon's orbit and suffered to fall, would, in a second of time, fall through a space equal to the sagitta of an arc of the Moon's orbit described in the same time; or, that the Moon brought down to the vicinity of the Earth, and a body there projected and describing a parabola would (the resistance of the air being supposed to be abstracted) be equally deflected from the tangents of their curves; the deflection being about sixteen feet.

The Proposition of the *Principia* to which we have referred is an easy instance of the application of the mathematical results, obtained by Newton in the preceding books, to the system of the universe. If PQ be an arc of the Moon's orbit described in $1''$, then,

$$RQ = \frac{(QP)^2}{2\,CP},$$

is the value of the *sagitta*, or of the *deflection* of the Moon from



the tangent of her orbit, or (by the second law of motion) of the space through which the Moon, or a heavy body at the Moon's distance, would fall in the same time. Now, according to the Law of Gravity, a space corresponding to RQ at the Earth's surface, would equal

$$RQ \times \left(\frac{CP}{CA}\right)^2, \text{ or (see \textit{Astron.} p. 95.)}$$

$$\frac{RQ}{(\text{☽'s horizontal parallax})^2}.$$

Hence, since $PQ = \dfrac{3.14159 \times 2\,CP \times 1''}{\text{☽'s period}}$, the space corresponding to RQ, at the Earth's surface, is equal

$$\frac{2\,.\,\text{⊕'s radius} \times (3.14159)^2}{(\text{☽'s parallax})^3.\,(\text{☽'s period})^2},$$

which, by computation is *nearly*[*] equal to the space fallen through by a heavy body at the Earth's surface.

This result most simply and clearly illustrated Newton's Theory. The deflection of the Moon, from the tangent of her

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[*] *Nearly* equal; for, the process needs several corrections.

First, the space RQ, the deflection of the Moon from the tangent, does not expound the whole effect of the Earth's attraction: for, by reason of the Sun's disturbing force, the Moon's gravity is diminished and by about its $\frac{1}{358}$th part; consequently, instead of RQ, $RQ \times \left(1 + \frac{1}{358}\right)$, expounds the Earth's attraction, and the corresponding space at the Earth's surface (see p. 24. l. 9.) ought to be

$$RQ \times \frac{359}{358} \times \frac{1}{(\text{☽'s parallax})^2}.$$

Secondly, the deflection, or the descent of the Moon in a given time, towards the Earth (whether or not we consider the Sun's disturbing force) does not arise solely from the Earth's attraction, but from the joint attractions of the Earth and Moon. For, according to the Principle of Gravity, every particle of matter attracts; the particles of the Moon, therefore, as well as those of the Earth. The *approach*, therefore, of the Moon to the Earth arises from their *mutual* action, and, consequently, that part of the approach which is due solely to the Earth is less than the whole in the proportion that the Earth's mass ($\oplus$) is less than the sum of the masses of the Earth and Moon ($\oplus + \text{☽}$): and, accordingly, the computed descent at the Earth's surface arising solely from the Earth's attraction, is now

$$RQ \times \frac{359}{358} \times \frac{\oplus}{\oplus + \text{☽}} \times \frac{1}{(\text{☽'s parallax})^2},$$

or, in numbers,

$$RQ \times \frac{359}{358} \cdot \frac{58.7}{59.7} \times \frac{1}{(\text{☽'s parallax})^2}.$$

This space, in order that Newton's Principle may be proved to be true, ought to equal the descent of a heavy body (ascertained by means of the pendulum) in one second of time; and it is to be observed, that, in strictness of principle, although they are far too minute to affect the computation, the two last corrections apply to the descent of a heavy body near the Earth's surface, or to its deflection from the tangent to a parabola: for such descent and deflection must be *less* (the question orbit, towards the Earth, and the fall of a body near the Earth's surface were shewn to be like effects. The mode of producing those effects formed no part of the enquiry; but, without absurdity, or the obtrusion of a theory, they might be said to proceed from the same cause, namely, the Earth's attraction. '*Vis quâ Luna,*' (says Newton in that remarkable Proposition * that has been just quoted) '*in suo orbe retinetur, illa ipsa est quam nos gravitatem dicere solemus.*'

This important point of the Moon's Gravitation being gained, there was opened to Newton an immense field for the farther

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is not about the degree) than it would be were the Sun away; and must be, in part, attributable to the *attraction* of the heavy body.

The third correction applies to the Moon's parallax. The parallax ought to be that which (see *Astronomy*, p. 315.) is called the *constant*, and which is the angle at the Moon subtended by that radius of the Earth which is drawn from the centre to a parallel, the square of the sine of which is $\frac{1}{3}$. In such latitude the centrifugal force $= \frac{2}{3}$ centrifugal force at the equator $= \frac{2}{3} \cdot \frac{\text{gravity}}{288}$. The descent ($s$), therefore, of a heavy body in this latitude does not expound the whole effect of gravity: but $s + \frac{s}{432}$ does: consequently, for the purpose of verifying Newton's Theory, or, more correctly, in order to prove that the Moon gravitates, this equality ought to subsist.

$$RQ \times \frac{359}{358} \cdot \frac{58.7}{59.7} \times \frac{1}{(\text{☽}\text{'s constant parallax})^2} = s\left(1 + \frac{1}{432}\right).$$

The corrections are almost as curious and important as the theory itself.

* We cannot, even at this distance of time, view without interest and anxiety, the momentous trial and test to which Newton thus subjected his system. Had that equality, which is stated at the end of the last Note, been found not to subsist, the System of Gravitation would have been as baseless as the Vortices of Descartes. We should have had no *Celestial Mechanics.* The *Principia* would have been reduced to its second Book: and Newton must then have gone down to posterity as an extraordinary man for his discoveries in Optics and pure Mathematics.

trial of his theory. Jupiter and his satellites, Saturn and his, immediately afforded him partial tests. He had little difficulty in shewing that the law, which connected the descent of a heavy body, and the Moon's deflection from the tangent of her orbit, connected, also, the *deflections* of the four satellites of Jupiter from the tangents of their respective orbits, and the deflections of Saturn's satellites from the tangents of their orbits. If the first satellite were said to *gravitate* to Jupiter, the second, third, and fourth gravitated also. If the first satellite of Saturn gravitated to its primary, the others did. This regards the Law of Gravitation. But it is not easy to prove, nor is it *proved* any where in the *Principia*, that if the Moon gravitated to the Earth by reason of, and in proportion to, the Earth's attractive particles, that the satellites of Jupiter and Saturn would *so* gravitate to their primaries. It was highly *probable* that they did so gravitate, but there could, by the ordinary methods, be no *proof* to that effect; since the masses of Jupiter and Saturn are not thereby assigned, except on the supposed truth of the *Principle* of Gravitation. The methods, therefore, could not establish the principle.

But this Principle of Gravitation, according to which every planet and satellite attracts in proportion to its mass, led to other considerations. It bore immediately on the demonstrations of Kepler's Laws, which demonstrations in the second and third Books of the *Principia* are, altogether, independent of attracting particles and masses; they relate to mere points. Would such demonstrations apply when, instead of points, a central Sun and a revolving planet were substituted? According to the Principle of Gravitation, if Jupiter gravitates to the Sun by virtue of the Sun's attractive matter, the Sun must gravitate to Jupiter by virtue of a like matter. So with regard to the Sun and Saturn, they must be drawn together by the sum of their separate attractions. These inferences, therefore, made the centripetal force greater than if the Sun solely acted: but they did not establish any alteration in the direction of the force; it would still remain centripetal. The demonstrations, therefore, of the equable description of areas, and of the elliptical form of the orbits of planets stood as they did before, when physical points represented the central and revolving bodies. But it would be otherwise with

Kepler's third Law. If the whole gravitations of Jupiter and Saturn to the Sun arose, respectively, from the masses of the Sun and Jupiter, and the masses of the Sun and Saturn, then the centripetal forces urging Jupiter and Saturn, at equal distances from the Sun, would be different; for, the masses of those two planets are unequal. That of Jupiter being the greater, the force by which he would seek the Sun's centre would be greater. In this case, therefore, the squares of the periodic times could not be to each other as the cubes of the mean distances: for, (see p. 22.) the equality of the centripetal force, at the same distance from the centre, is an essential condition in the demonstration of that analogy.

The demonstration, therefore, of Kepler's third Law requires*, on the preceding principles, some slight modification: but the law, as it appears from observation, is *very nearly* true: the immediate inference from which is, the minuteness of the mass of the revolving body, whether it be Venus or Jupiter, compared with the Sun's.

But the modification, just alluded to, is altogether insignificant when compared with other consequences that flowed in on like extensions of the Principle of Gravitation. For, if Jupiter and Saturn, by reason of their matter, attracted the Sun, they would for the same cause, attract each other. In conjunction, Saturn would draw both the Sun and Jupiter towards him, but the former less than the latter. Jupiter's gravity, therefore, to the Sun would be diminished; but, the direction of the diminishing force conspiring with Jupiter's gravity, the resulting or com-

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[*] Very simple considerations are sufficient to shew that the relation of the dimensions of the orbit to the time of describing it, cannot be independent of the mass of the revolving body. Suppose (and this is the condition in Newton's demonstration of the third Section) the revolving body to be a point, and, moreover, the orbit to be circular. If instead of a point we substitute a mass, then by reason of the increased attraction, an orbit interior to the circle must be described in a shorter time, and having the sum of its least and greatest distances less than the diameter of the circle. If the mass were Jupiter's the orbit would be more contracted within the circle than if the mass were Mars's.

pounded force would be still centripetal, and, consequently, in such position of the planets, the equable description of Areas might, notwithstanding the alteration of force, still subsist. But, the moment after conjunction, the two forces, Jupiter's gravity, and Saturn's diminishing or disturbing force, would not conspire in direction, and a *perturbation* of motion would necessarily ensue.

The perturbation would principally affect the equable description of areas: which equable description depends essentially (see *Principia*, Sect. II. Prop. 1.) on the uniformity of motion in the direction of the tangent. Now, when Jupiter has either quitted or not reached the line of conjunction, part of Saturn's disturbing force must act in the direction of the tangent to Jupiter's orbit and alter the velocity. It would add to the velocity, if Jupiter were approaching conjunction, and diminish it, after he had left it. In the former case, the small line expounding the linear velocity being made less, the area described would become less, or would be *retarded*. In the latter case, the exponent or measure of the velocity being greater, the area would be so also, or, technically speaking, be *accelerated*. But, it is plain, Kepler's Law is equally transgressed whether it be by an *acceleration* or by a *retardation* of areas.

We have here, then, by the necessary consequence of the Principle of Gravitation, a violation of Kepler's Laws, and an inlet, into the elliptical system, afforded to a series of perturbations: for, it is contrary to all probability, that the perturbation, which has just been mentioned, would enter singly.

It is, indeed, easy to see, that other perturbations would enter with it. That Law of Kepler's, which establishes the elliptical form of the orbits of planets, would be transgressed: for the two essential conditions in its demonstration, are (see *Principia*, Sect. III. Prop. 12, &c.) that the force should be centripetal, and should vary according to the inverse square of the distance.

Now, as we have already seen in l. 11, &c., a part of Saturn's disturbing force would act in the direction of a tangent to Jupiter's orbit, and it is not likely that such force, whatever its law, would not disturb the description of an ellipse: since that description would take take place were the disturbing force away. But, besides, the difference of forces, by which Saturn draws the Sun and Jupiter towards him, cannot be entirely resolved in the direction of a tangent to Jupiter's orbit. Some resolved part of such difference must act in the direction of the radius vector of Jupiter's orbit. It, therefore, conspires with Jupiter's gravity, but it does not vary as that gravity does. The resulting force, therefore, would not so vary: and, if it be not an undeniable consequence, that an ellipse cannot be described by these two forces, the last compounded one, and the former tangential, yet the demonstration, just referred to, (p. 29. l. 30, &c.) of the description of such a curve, is totally inapplicable to the new conditions.

On general grounds, then, if Jupiter, attracted solely by the Sun, describes an ellipse, it is, at the least, improbable he should describe such a curve when disturbed by Saturn's attraction. There is, indeed, no palpable absurdity in supposing that Jupiter should deviate from one ellipse into another by the agency of external forces: but the circumstance is not to be presumed, and is, in fact, contradicted by calculation.

We have already seen (see p. 28.) from one point of view, that Kepler's third Law is restricted, and after what manner it is restricted. None, therefore, of the laws can subsist, with entire truth, when more than two bodies compose the system and each body attracts the others proportionally to its mass. When abstraction is made of the masses of the planets and of all attraction saving that of the central body, the Laws of Kepler are strictly true: they are strictly true, therefore, in an ideal system, and very nearly true in the system of Nature.

But then comes this dilemma: if Kepler's Laws are only nearly true, their demonstration, on restricted or partial conditions, can be no absolute proof of the truth of the Principle and Law of Gravitation. The proof, therefore, must be sought for elsewhere, or in farther investigation. Instead of proving that Jupiter describes an exact ellipse, it is necessary to find how much, in his course, he deviates from one. Instead of proving the theory of gravity from the equality of areas described by the Moon, it becomes necessary to attempt that end by means of their *acceleration*.

We may conjecture that, with considerations somewhat like the preceding, Newton viewed the wide field of investigation that was opened to him, on passing the limits of the Elliptic System. Every particle of matter attracting, parts of the same system, and of the same body, would be unequally attracted, and, in technical language, mutual *perturbation* would ensue. 'Graves sunt (says the great Author * of the *Principia*) planetæ omnes in se mutuo per Cor. 1. et 2. Et hinc Jupiter et Saturnus prope conjunctionem se invicem attrahendo, sensibiliter perturbant motus mutuos: Sol perturbat motus Lunares, Sol et Luna perturbant mare nostrum.'

We may still farther conjecture Newton to have been embarrassed by the great variety of objects that presented themselves, each of which would serve, in some degree, as a test and touchstone of his hypothesis. Ought the greatest or the least perturbation to be first selected? The instance of the Moon or of Mars? He had the option of beginning his research, either with the planet that the least transgressed, or with that which most transgressed the laws of elliptic motion. If he began with Mars, its irregularities would require no very operose calculations: but, then, to balance this, those irregularities, as phenomena of observation, were not distinctly made out, were very minute, and, in quantity, not very different from the errors of observation. Besides this, the irregularities, were, probably, a blended effect of the disturbing forces of the Earth and Jupiter. But the Moon's inequalities (the greater of them, at least,) were, when Newton began his researches, distinctly known, were large in quantity, and, on the Principle of Gravity, could only proceed from so great a body as the Sun.

To the selecting, however, of these irregularities as tests of his system, there existed an impediment, almost an enormous one in

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* Lib. III. Prop. 5.

Newton's time, and which even now subsists: namely, the extreme difficulty of computing them. But, great as it was, Newton found means of overcoming it, and from the *Lunar Inequalities* derived his second series * of proofs of the truth of the Theory of Gravity.

These proofs, as it has been already said, differed, in one respect, essentially from the former: they were founded on the *deviations* from the Elliptical system, the former on the system itself. Newton's Theory might be true if a planet described an ellipse nearly: it could not be true, if it described an exact ellipse.

Newton, in his process of proofs, from the exact ellipse went at once to the most irregular orbit. He did not pass through any gradations of slightly disturbed ellipses; such as the orbits of most of the planets are. It is the drift of his investigations in the eleventh Section of the first Book, and in the third Book, to account for the large irregularities of the Moon's orbit from the Sun's disturbing force: to shew that such irregularities or inequalities are as certain a consequence of that disturbing force, as the elliptical orbit of Mars is of the Sun's centripetal force.

This *accounting for* the Lunar inequalities, by the agreement of their observed and computed quantities, was then, as it is now, a most arduous undertaking; but, it is not to be surmised, that Newton designedly passed over the most simple proofs of his Theory in order to arrive, the sooner, at the most elaborate. The more probable reason is, as we have already hinted, that the planetary inequalities, although consequences of the same kind, and from like causes, as the Lunar, were judged by Newton to be too minute either for illustrating or for confirming his theory. They certainly were not distinctly noted at the time of the publication of the *Principia*: and it is not an unfair inference, from what* has just been quoted (see p. xxxi.)† that its Author supposed the perturbations of Jupiter and Saturn, *except in conjunction*, to be very inconsiderable.

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* The history of these matters is not easily to be made out, but we may *conjecture* that the second series of proofs less powerfully impelled Newton's contemporaries to the belief of his system than the first. For, the extreme simplicity of the Elliptical system operating on the natural fondness of the mind for such quality must have created it many adherents. But it was otherwise with the system of Perturbations; that was devoid of simplicity: its results were not uniform, and, which alone would lessen the number of its partisans, not to be reached except by investigations beyond ordinary research.

He knew the case to be quite different with the Lunar inequalities.

The term *Inequality* implies a departure from something either previously equable and regular, or that would be so, were it not for the intervention of certain causes. The causes, by a corresponding technical denomination, are called *Disturbing*. If the Moon described an ellipse there could be no disturbing causes: and there would have been no impropriety in calling the elliptical motion regular and exempt from *inequality*. But the usage of terms is somewhat different, and the technical language employed, on this occasion, has been accommodated to the views which Astronomers have chosen to take of this subject. They have considered circular motion to be, as undoubtedly it is, more simple and regular than the elliptical, and the latter to differ from the former, or to be *unequal* to it, by a certain *inequality*. The origin of *Inequalities* has thus been thrown farther back, and, what is more to be observed, been made, in one instance, independent of disturbing causes; for, the *elliptic*, which is sometimes called the *first Inequality*, is so independent; and, on that account, is naturally of a different class from that to which the other Lunar inequalities arising from the Sun's disturbing force belong. It is, however, as it has been mentioned, conventionally included in that class.

Kepler, by means of his *Problem*, found a planet's elliptical place. Newton had to find the difference between the Moon's true and her elliptical place; the difference depending on the Sun's disturbing force. The *difference* had been already partially assigned and parcelled out by Tycho Brahé, and other Astronomers prior to Newton, into several *Inequalities* distinguished by the names of *Variation*, *Evection*, and *Annual Equation*. As this difference of place, which is an effect, had been divided and distinguished into parts, so Newton resolved the cause, which is the Sun's disturbing force, into several parts or *modifications*; and, more than this, he assigned to some of these modifications, as to their special causes, certain of the inequalities: for instance, to that part of the disturbing force which acts in the direction of the tangent to the Moon's orbit he attributed the *Variation:* and the *Annual Equation* to a modification of that part of the disturbing force which is resolved in the direction of the radius.

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* 'Et hinc Jupiter et Saturnus prope conjunctionem se invicem attrahendo sensibiliter perturbant motus mutuos.'

† The preceding references to the Preface should have been like this, and in Roman numerals.

In assigning to different modifications of the disturbing force of the Sun the several Lunar inequalities, there are two methods pursued by Newton; the one popular, the other strict and scientific.

The first method is followed in the eleventh Section of his *Principia*. In that the Author shews the origin of the Sun's disturbing force, and expresses its value. But he does not thence deduce any exact results, such as might be compared with those of observation. He has rather chosen to usher in his Theory by explaining the general nature and character of the effects produced by the Sun's disturbing force; after what manner, for instance, the curvature of the Moon's orbit, the equable description of areas, the positions of the nodes and apsides are affected or *disturbed* by it.

He expounds the principles for computing the perturbations, but does not there compute them. Indeed, the character given by the great Author himself of the composition * of part of his third Book would not have ill-suited that of the eleventh Section. Its explanations are of a general nature. They do not, therefore, prove, they only render probable the Principle and Law of Gravitation. The proofs, such as the subject admits of were reserved for subsequent investigations, and are given in the third Book of the *Principia*.

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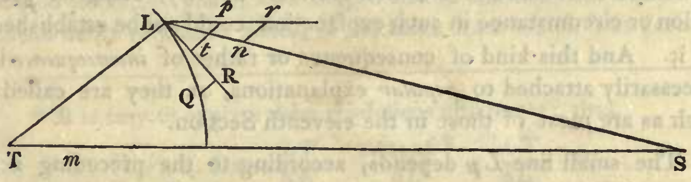
* 'Methodo populari, ut a pluribus legeretur.'

The *popular* explanations were judiciously made to precede the strict calculations. The operose computation of the *Variation*, of the mean regression of the nodes, &c. on a *new* principle, and by the aid of a *new* calculus, would have been ill adapted to an age scarcely freed from the school philosophy and the system of Descartes.

But the explanations of the Lunar and other inequalities in the eleventh Section, are, as it has been already remarked, far from proving rigidly, either the Principle or the Law of Gravity; and it may easily be shewn, they do not prove the latter. For, admitting the *principle*, there would exist a tangential disturbing force, accelerating the Moon in some situations, and retarding her in others, and producing an inequality, like, in its general character, to the *Variation*, if the Law of Gravity, instead of the inverse square, were the inverse cube, or some intermediate inverse power. For instance, a body at S, whatever the law of force, would attract L and T unequally, if L and T should be at unequal distances from S: if the force should vary in any inverse power of the distance, L, being nearter to S than T, would be more attracted: suppose Tm, Ln to expound the *gravitations* of T and L to S, and resolve Ln ($> Tm$) into Lr parallel to TS, and rn parallel to LT. Take $pr = Tm$, then, the difference of forces, by which T and L are drawn towards S in the parallel directions TS, Lr, would equal Lp ($= Lr - Tm$).



Again, resolve Lp into Lt, pt, pt being parallel to LT: we have now then the body L attracted to T by a centripetal force, expounded by QR, a force rn augmenting it, and therefore being itself centripetal, a force pt diminishing it, and a tangential force Lt urging L in the direction LR, or, in other words, increasing its velocity. Now, as it is plain, this tangential force Lt, as well as the other forces rn, pt, arise solely from the inequality of attractions of T and L towards

S, and are altogether independent of the law of attraction which may be either as $\frac{1}{D^2}$, or $\frac{1}{D^{\frac{3}{2}}}$, or $\frac{1}{D^{\frac{5}{2}}}$, or $\frac{1}{D^3}$, D being the distance between S and the attracted body.

We have supposed the body L to be moving in the direction LR, since we have supposed the tangential force Lt to accelerate its motion: a like force would retard L after it had passed the line of syzygies. In that line the tangential force would be nothing, as it would be also in some point between syzygies and quadratures.

A tangential force, therefore, such as we have described, would, in a general way, and to a certain extent, explain the *acceleration* and *retardation* of the Moon's areas, or in other words, the Moon's *Variation*. But the *Law* of Gravity, being no condition or circumstance in such explanation, could not be established by it. And this kind of consequence or rather of *inconsequence* is necessarily attached to *popular* explanations, as they are called; such as are most of those in the eleventh Section.

The small line Lp depends, according to the preceding account on T and L being unequally attracted by S, which on that account is called the *Disturbing* body: disturbing the equality of areas and the curve, which, were it abstracted, L would describe round T. Both these perturbations, as to their general nature, would take place, if the force by which S draws L and T, were not according to the inverse square of the distance. That, however, is the Law which Newton, in his eleventh Section, and in the third Book of the *Principia* means to establish: and which, indeed, the more minutely and scrupulously the planetary phenomena are examined, seems sufficient to account for them.

According to that law, then

$$Tm = \frac{\text{mass of } S}{ST^2}, \quad Ln = \frac{\text{mass of } S}{SL^2}.$$

and $QR \left(= \frac{\text{mass of } T}{ST^2} \right)$ drawn parallel to TL, representing the force by which L is drawn by T towards T, the whole centripetal force urging L is

$$QR + rn - pt,$$

and the tangential force, or, if the orbit be circular, the force perpendicular to TL, is Lt[*]: and such, although somewhat differently deduced and stated, are Newton's resolutions of force in his eleventh Section.

To like resolutions, or, as they may be called, *Modifications* of the Sun's disturbing force, Newton assigned, as to their special causes, certain of the Lunar inequalities. Of these the most notable is the *Variation*, as being the deviation from the most simple of Kepler's Laws, namely, the equable description of areas. Newton attributed this inequality, principally, to the tangential force (Lt). But, as we have seen, such a force would subsist, and would *accelerate* and *retard* the description of areas, or (stating the infringement of Kepler's Law in other terms) would make the planet now before and at another time behind its mean or its *elliptical*[†] place, if the force from which it is derived,

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[*] It is easy to express these resolutions differently: thus,

$$rn = Ln \times \frac{LT}{SL} = \frac{\text{mass of } S}{SL^2} \times \frac{LT}{SL},$$

$$pt = Lp \times \cos. LTS$$

$$= (Lr - pr) \cos. LTS$$

$$= \left(Ln \, . \, \frac{ST}{SL} - Tm\right) \cos. LTS$$

$$= \text{mass of } S \times \left(\frac{ST}{SL^3} - \frac{1}{ST^2}\right) \cos. LTS,$$

the whole centripetal force, therefore, see l. 1. of the text, is

$$\frac{\text{mass of } T}{LT^2} + \text{mass of } S \times \left[\frac{LT}{SL^3} - \left(\frac{ST}{SL^3} - \frac{1}{ST^2}\right) \cos. LTS\right],$$

and the tangential force (Lt) is

$$\text{mass of } S \left(\frac{ST}{SL^3} - \frac{1}{ST^2}\right) \sin. LTS.$$

[†] *Elliptical*, or found in the ellipse by *correcting* the mean place, on account of the *first*, or elliptical inequality, and by means of Kepler's Problem. See *Astronomy*, Chap. XVIII.

namely, that by which S draws L and T should not vary according to the inverse square of the distance. In order, then, to ascertain whether the latter law of the variation of the force, or any other law, be the true one, a closer examination of results must be instituted; in fact, the inequality itself must be computed, and in his third Book the Variation itself (see Prop. XXVI, XXIX.) is computed from those expressions of the force which are given in the Note of the preceding page, and which suppose the law of the force to be that of the inverse square of the distance. The agreement of the computed and observed variation proves the law of the force to have been rightly assumed, and, as far as a single instance can go, proves the truth of that law.

It is so with other popular explanations and strict computations of other Lunar inequalities: the former render probable the Theory of Gravitation, the latter, if their results agree with those of observation, confirm it.

The diagram which we have already introduced may serve to explain after what manner another Lunar inequality, may, on Newton's principles, be assigned, (as to its special cause), to a modification of a resolved part of the disturbing force.

The part of the disturbing force which acts in the direction of the radius vector LT is (see p. xxxvi.)

$$r n - p t,$$

which sometimes augments, at other times diminishes the centripetal force RQ that arises from the central body at T: or which, since we are speaking of the Lunar Theory, so affects the Moon's gravity to the Earth. The latter, the diminishing effect, predominates: so that, during one synodic revolution, there takes place what may be called, a *Mean Diminution* of the Moon's Gravity. Hence ensues an augmentation of the Moon's periodic time beyond what would have been its value, had there been no disturbing force; and, the greater the mean diminution of gravity, the greater the augmentation of period. Now the diminution we are speaking of, is the result, or mean, of the several diminutions and augmentations that happen during one synodic period, the Earth's distance being supposed to remain the same. If the distance be changed, the result, or *mean*, will be changed *.

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* The computation is easily given.

Let m be the mass of the Sun
r' the radius of his orbit,
r the radius of the Lunar orbit,
$y = SL$, and $\theta = \angle LTS$, then, see p. xxxvii.

$$rn - pt = \frac{mr}{y^3} - \left(\frac{mr'}{y^3} - \frac{m}{r'^2}\right)\cos.\,\theta.$$

But $y = \sqrt{(r'^2 + r^2 - 2rr'\cos.\,\theta)}$,
and, r' being very large relatively to r,

$$\frac{1}{y^3} = \frac{1}{r'^3} + \frac{3r}{r'^4}\cos.\,\theta, \text{ nearly};$$

$$\therefore\ rn - pt = -\frac{mr}{2r'^3} - \frac{3mr}{2r'^3}\cos.\,2\theta.$$

Now the sum of the disturbing forces in the direction of the radius is the sum of the several parts on the right-hand side of the equation taken for every point in the circle, from 0° to 360°. The sum of all the $\frac{-mr}{2r'^3}$, is $\frac{-mr}{2r'^3}$ × circumference. With regard to the second part, since $\cos.\,2(90^\circ + \theta) = -\cos.\,2\theta$, whatever be the value of θ, there must be as many terms of the form $\frac{3mr}{2r'^3}\cos.\,2\theta$, as of the form $-\frac{3mr}{2r'^3}\cos.\,2\theta$, the corresponding terms being respectively distant from each other by 90°. The sum, therefore, of all the terms $-\frac{3mr}{2r'^3}\cos.\,2\theta$, θ being taken of every value from 0 to 360°, must be nothing, the corresponding positive and negative terms destroying each other. The sum, therefore, of all the $rn - pt$ is $\frac{-mr}{2r'^3}$ × circumference: consequently the mean effect will be

$$-\frac{mr}{2r'^3} \times \frac{\text{circumference}}{\text{circumference}},\ \text{or } -\frac{mr}{2r'^3},$$

which may be supposed constant during one synodic revolution: but which will vary when r' the Sun's distance from the Earth varies.

Every month, therefore, it will be changed. It will be increased by the Earth's approach to the Sun, and, consequently, will be greater in Winter than in Summer. Now these results which flow immediately from that modification of the resolved part of the disturbing force which acts in the direction of the radius, resemble the properties of the *annual equation* (see *Astron.* pp. 328, &c.): and, therefore, to go no farther, that irregularity in the Moon's motion, is also probably caused by the Sun's disturbing force: and if exactly accounted for, that is, by the agreement of its computed with its observed quantity, would additionally confirm the Theory of Gravity *.

It was thus that Newton shewed that the Sun's disturbing force (a necessary consequence of the Theory of Gravity), if not their real cause, would at least account for two of the Lunar inequalities: the *Variation* discovered by Tycho Brahé, and the *Annual Equation* discovered by Kepler from the former Astronomer's observations.

But there existed, of much older date, and discovered by Hipparchus and Ptolemy, another considerable Lunar inequality called the *Evection:* and this, also, Newton shewed to be explicable by the theory of Gravity.

In the explanation of this inequality the eccentricity of the Moon's orbit is an essential condition; which is not the case with the explanation of the two former inequalities. The *Evection* arises from the variation of the eccentricity of the orbit: and, the variation of the eccentricity from a modification of the general effect of a resolved part of the disturbing force dependent on the position of the apsides.

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* Hisce motuum Lunarium computationibus ostendere volui, quod motus Lunares, per Theoriam Gravitatis, a *causis suis* computari possint. Per eandem Theoriam inveni præterea quod Æquatio annua medii motus Lunæ oriatur a variâ dilatione orbis Lunæ per vim Solis, juxta, Cor. 6. Prop. LXVI. Lib. 1. Scholium, Lib. 14.

The *general* effect of that part of the disturbing force which acts in the direction of the radius is to augment the Moon's Gravity in quadratures, and to diminish it in syzygies. Its *peculiar* effect when the apsides lie in quadratures (which is a modification of its general effect) is, since it varies as the distance, to augment the Perigean Gravity, which is the greatest, by the least quantity, and to augment the Apogean Gravity, which is the least, by the greatest quantity. Its peculiar effect, then, in this position of the apsides, is to make the ratio between the Perigean and Apogean Gravities less than that of the inverse square of the distances, and thereby to render the orbit less eccentric. When the apsides lie in syzygies, the peculiar effect of the resolved part of the disturbing force is to render the orbit more eccentric, by rendering the ratio between the Perigean and Apogean Gravities greater than that of the inverse square of the least and greatest distances; because, in this position of the axis major, the Perigean Gravity is *diminished* by the least and the Apogean by the greatest quantity. Now the eccentricity being, as it is in the first position of the apsides, diminished, the *Equation of the Centre* (see *Astron.* p. 202.) will be necessarily diminished; and in the second position the eccentricity and the *Equation of the Centre* will be increased. In intermediate positions the effect will be a blended one.

But, whatever the position, the variations of the Equation of the centre as consequences deduced from the disturbing force, correspond to the observed and registered properties of the *Evection.* The former, therefore, will account for the latter. The inequality is an effect of which that modification of the Sun's disturbing force just spoken of seems to be the adequate cause (see *Astron.* pp. 325, &c.).

It was nearly after the preceding manner that Newton, in his eleventh Section, explained the three principal Lunar inequalities, by his Theory of Gravity; and, by such explanations, added new proofs of the truth of that theory: the proofs being, as it has been already mentioned, of a different kind and of a higher scale than those which had been derived from the establishment of two of Kepler's Laws.

But the proofs, derived from the same sources, are not so strict as we find them in other parts of his Work. Those of the eleventh Section were not even proposed, as their Author himself informs us *, as perfect ones. They do not establish (and this is a point more than once insisted on) the Law of Gravity to be exactly according to the inverse square of the distance. Most of the reasonings of the eleventh Section hold good, if the Law, as Clairaut proposed in a memorable instance (see *Mem. Acad.* 1745.) instead of being as it is, should only be nearly so, and should be expressed by a formula of two terms.

After explaining for what reasons, founded on the Theory of Gravity, the Moon would deviate † from the laws of elliptical motion, Newton proceeded to other proofs of that theory: or rather to the explanations of other phenomena which it afforded. The inequalities which he had traced to the Sun's disturbing force, as their source, were *periodical:* admitting of alternate increase and decrease, and, at the completions of their periods, returning to those values or magnitudes which they had at the beginnings. But the Moon was subject to other inequalities; such as affected not her position in the orbit, but the dimensions of that orbit and its position in space. To the same source, the Sun's disturbing force (which, as it has been often said, itself springs from the principle of gravity) he traced these latter inequalities. On just principles, and, completely, he explained the motion of the Nodes of the Lunar Orbit, their mean *Regression,* and the change of the orbit's inclination; but imperfectly, since on partial principles, the *Progression* of the apogee. After having explained the Regression of the Nodes he beautifully applies the result of such explanation to the precession of the equinoxes: and then passes on to shew in what manner the phenomena of the Tides are, on the Theory of Gravity, explicable from their causes, which are the attractive, or disturbing forces, of the Sun and Moon.

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* 'Quæ ad motus Lunares spectant (imperfecta cum sint) in Corollariis Propositionis LXVI, simul complexus sum, ne singula methodo prolixiore quam pro rei dignitate proponere et sigillatim demonstrare tenerer, et seriem reliquarum propositionum interrumpere.' *Auctoris Præfatio.*

† It seems remarkable that Newton should have forborn announcing in distinct terms that his Theory afforded an explanation of those principal Lunar inequalities which Astronomers had noted and distinguished by technical denominations. Such an announcing would have drawn the attention of the curious most forcibly to his Theory.

All these phenomena, the tides, the precession of the equinoxes, the periodical inequalities of the Moon, the inequalities of the *elements* of its orbit (for the inclination, the eccentricity, the positions of the node and apogee, are technically so called) are familiarly explained (*methodo populari*) in the eleventh Section. The succeeding Section is occupied with different investigations.

But, although in kind different, they like those of the Lunar inequalities, date their rise from the same source, the principle of Gravitation. According to that principle not only all the bodies of the system, the Sun and planets, separately attract as masses, but the component parts or particles of each mass. Now in some of the Sections previous to the twelfth, which treat of Kepler's Laws and the deviations from those laws caused by disturbing forces, the central, revolving and disturbing bodies are considered as merely physical points. No account is made of their figure. But such are not the conditions in nature. The Earth, which, as the central body, *attracts* the Moon, and the Sun which as the *third* and external body *disturbs* the Moon (disturbs it by attracting it more or less than it does the Earth) are both endowed with volume and figure; and then comes this question; if each particle of the Earth attracts the Moon, as it must do according to the Principle of Gravitation, to what point in the Earth will the result of the several attractions be directed? If it should be the centre of the Earth, then what had been demonstrated in the preceding Sections relative to Kepler's Laws would still hold good. But if not, then the previous results, in order to be adapted to the real circumstances of nature, would, at the least, require some correction.

To this subject of enquiry Newton directs his attention in the twelfth Section. He therein demonstrates that, the law of attraction being the inverse square of the distance, spheres attract just as if all their matter were condensed into their centres. Accordingly, if the planets were spherical bodies, the results which had been proved to belong to merely central points might be transferred to them; and they may be transferred with very little inaccuracy, since the planets, although not exactly spherical bodies, are nearly so.

In truth, however, spherical planets, like elliptical orbits, belong to an ideal system and have no place in nature. Their orbits are only *nearly* elliptical and their figures *nearly* spherical. If we assume the former to be ellipses and the latter spheres, it is for the purpose of conveniently commencing our processes: of making, by such first steps, approximations towards remoter results.

But it must not be supposed, from what has just been said, that the inequalities of motion of the revolving body caused by the *oblateness* of figure in the central are at all comparable, either for their magnitude or number, to those that are caused by an external disturbing body. The Lunar inequality caused by the small *ellipticity** of the terrestrial spheroid may be corrected by one *equation*, the maximum of which does not exceed seven seconds: whereas there are nearly thirty *equations* which it is necessary to make account of, in correcting the deviation from elliptical motion caused by the Sun's disturbing force.

The result, therefore, of the seventy-first Proposition (Sect. 12. Book I.) is applicable, with very little error, to the Lunar theory. The inequality that would, virtually, be neglected by such application is less, as we have seen, than seven seconds; a quantity less than the error of those observations which were made in Newton's time.

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* By theory the oblateness of the Earth causes an inequality in the Moon's motion: and, consequently, which is very curious, we may from the Moon's motion determine the ellipticity or the degree of the Earth's oblateness.

We have now enumerated and slightly described the series of explanations which Newton, by means of his Theory of Gravity, has given of the planetary phenomena; which explanations, under another point of view, are so many proofs of the truths of that theory. They are all to be found in the *Principia*. But their matter, although by far its most important constituent, occupies less than the half of that extraordinary Work. If it were permitted not to approve of every thing that Newton has done, we might regret that the great argument, by which the Principle and Law of Gravitation are established, should be so mixed up and interrupted, as it is, with foreign matter; more foreign now, indeed, than it then seemed to be, when other systems, hardly out of vogue, were to be put aside or asunder in order to make room for a new one. Still the business of refutation and other matters interrupted the main course of argument. The progress of Physical Astronomy was impeded by the very abundance of the riches of the *Principia*. Had its contents not much exceeded or not much varied from what Newton originally communicated to the Royal Society*: or if it had contained part of the second Section, the third, seventh, ninth, eleventh, twelfth Sections of the first Book and the third Book, it would probably have had more numerous readers.

But be this as it may. Still the *Principia* is the most prodigious Work in mathematical Philosophy that was ever produced†. To estimate its merit we must view Science as its Author found and as he left it. He did not merely add to or beautify a system. Newton's merit was more than that of having left marble what he found brick. For he laid the very foundations of Physical Astronomy, and furnished the materials and the means of putting them together.

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* 'Quippe cum demonstratam a me figuram orbium cœlestium impetraverat, rogare non destitit ut eandem cum Societate Regali communicarem.' *Auctoris Præfatio.*

† 'And it may be justly said, that so many and so valuable philosophical truths, as are herein discovered and put past dispute, were never yet owing to the capacity and industry of any one man.' *Halley's notice of the publication of the Principia:* see *Phil. Trans.* 1687, No. 186.

The Author of the *Principia*, although he lived in an age rich in men of genius, far outstepped his contemporaries: and from this circumstance, or the abstruse matter of his Work, or its manner, it happened that nearly sixty years elapsed from the first publication of the *Principia* before any material researches, on its principles, were made in Physical Astronomy. Clairaut began to make such about the year 1743. And then so limited seems to have been the study of Physical Astronomy, that the Author of the *Histoire** prefixed to that volume of the Paris Acts which contains Clairaut's Memoir, thinks it expedient to illustrate its subject, and to explain, by instances, the most easy and familiar, after what manner Newton conceived a planet to describe an elliptical orbit round the Sun.

The researches of Clairaut, of which we have just spoken, were given in the *Memoirs of the Academy of Sciences* at Paris for the year 1743. And the Memoir was entitled '*De l'Orbite de la Lune dans le Systeme Newtonien.*' We may discern in it the ground-work of that method which the Author and other foreign mathematicians afterwards used in their researches in Physical Astronomy. The method is widely different from Newton's; and the object of research is shortly stated to consist in the solution of the *Problem of the Three Bodies.*

By means of the opposite diagram we may easily explain the conditions and object of that problem.

When L revolves round T and is solely acted on by a centripetal force, directed from L towards T as a centre, the curve described by L and the laws of its motion are the objects of solution in the *Problem of Two Bodies;* and, as we have seen, Kepler's Laws are strictly observed in such system.

A third body placed at S, but not so distant as to attract T and L equally and in parallel directions, prevents the above laws from being observed: it *disturbs* what would take place were it away. The form of the orbit now described by L and the laws of its motion under the change of circumstances, or the deviations from the former orbit and laws (for each enquiry is, in fact, the same) are the objects of solution in the *Problem of the Three Bodies*.

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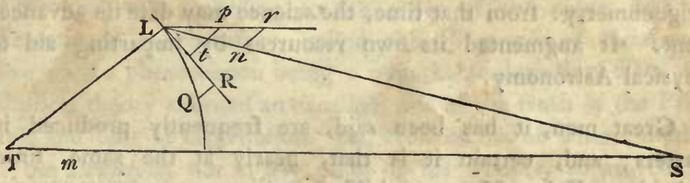
* The Histoire contains the abstract and brief explanation of the subjects treated of in the Memoirs and body of the Work.

The conditions are: a body L moving with a given velocity



in the direction LR, and acted on by two forces: one in the direction of the radius LT, the other in the direction of the tangent LR: the first compounded of the attracting force of T, and of the resolved parts of the disturbing force of S, and (see p. xxxvii.) equal to

$$QR + rn - pt,$$

the second consisting solely of a resolved part of the disturbing force of S, and equal to Lt.

Such are the objects and conditions of this famous problem, which are, in substance, the same as in the investigations that relate, in the *Principia*, to the Lunar orbit. But Clairaut uses *means* of solution far different from those of Newton.

The means used by Clairaut are differential equations of motion, in which the analytical expressions of the above-mentioned forces being substituted, and the equations solved, all that related to the Lunar orbit (if that were the object of enquiry) would be known.

We allude, in what has latterly been said, rather to a second than to that first Memoir of Clairaut's with which he began his researches in Physical Astronomy. This second Memoir was for the year 1745, and is incomparably more ingenious and profound than the one of 1743. It contains that remarkable differential equation of the second order and its integration (see p. 93. of the present Work), to which the labours of eminent mathematicians during seventy years have added scarcely any thing. The original mode is still adopted.

In the same Memoir we also find some of the first subsidiary or collateral uses, as they may be called, of the formulæ of Trigonometry: from that time, the science may date its advancement. It augmented its own resources by imparting aid to Physical Astronomy.

Great men, it has been said, are frequently produced in clusters; and, certain it is that, nearly at the same time, Dalembert, Euler, Mayer, and Thomas Simpson, contemporaries of Clairaut, began, like him, their researches in Physical Astronomy and on the same plan. In all these researches the first common step was a differential equation such as we have just referred to; and in all, although there are gradations of difference, its approximate integration was reached by means of the subsidiary formulæ of Trigonometry. These latter, in three of the Treatises, are made to precede the main investigation.

It has been already observed that the method of solution used by Clairaut and his contemporary mathematicians, was novel and altogether different from Newton's. The only point in which they agreed was in the expressions for the disturbing forces, which could not well be different. The Theory of Gravity, then, must needs receive great confirmation, if its results obtained by methods so different should agree with one another, and with phenomena: or, if any new results obtained by the new methods should be found to coincide with observation. Now both these things happened. Clairaut, by his peculiar method, deduced several Lunar inequalities, their coefficients and arguments, and found them to agree with Newton's results and with observation: besides this, he computed on Newton's principles (what Newton himself had not done) but by his own method, the *Progression of the Lunar Apogee*. Now this being a very important circumstance in the History of Physical Astronomy, deserves some farther consideration.

The place or longitude of the apogee is (see *Astronomy*, Chap. XXXI.) one of the *elements* of the Lunar orbit. If the Moon described an exact ellipse, which it would do but for the Sun's attraction, the place of the apogee would be fixed; but, as observation shews, it is variable, and according to Newton's Theory, by reason of the Sun's force, called under such circumstances of its action, a *disturbing* force. The mean annual motion of the apogee is according to the order of the signs, and the phenomenon is, therefore, called the *Progression of the Apogee.* Now such a phenomenon being a remarkable deviation from the elliptical theory afforded an excellent test of the truth of the Principle and Law of Gravitation. But Newton has no where, neither in the *Principia,* nor in any other of his Works, so proved his Theory.

He has merely proved that one resolved part of the Sun's disturbing force, namely, that which acts in the direction of the radius, will produce a *progression of the apogee:* but he has made no account of the tangential disturbing force, nor shewn the agreement between the results of his Theory and those of observation. Towards this important point, therefore, unattained to by Newton, Clairaut, as it might naturally be expected, directed the efforts of his new method; and its first result was a quantity only half that quantity of the progression which observations gave. And, in this result, Dalembert and Euler who were prosecuting like researches, coincided. An anomalous result of such magnitude occasioned doubts to be entertained of the truth of Newton's Law of Gravity.

The argument, drawn from this instance of the Lunar apogee, appeared so strong against Newton's Law of Gravity, that Clairaut proposed a new one, to be expressed by a formula of two terms such as

$$\frac{a}{(\text{dist.})^2} + \frac{b}{(\text{dist.})^4}:$$

the first of which was to expound the *old law,* the second a small addition to be made to it.

But this alteration was abandoned almost nearly as soon as it was projected. For Clairaut, on re-examining his method, found the source of his error. The method was one of approximation. Now it is not essential to such methods that the first steps should conduct us to results very near the truth: in that which Clairaut employed there was this peculiarity, that it conducted only half way. By a repetition, however, of the process, which every method of approximation under some shape or other requires, the next step was nearly as large as the first, so that then it was found that the computed *progression* was nearly equal to the observed.

It so happened that just as Clairaut announced his *discovery* (for such, considering the agitation and importance of the question, was nearly its nature) Euler and Dalembert had succeeded in correcting their processes. The *Progression* of the Lunar apogee was thus explained by its cause, on the Theory of Gravity. And this explanation was the first great addition or confirmation which that theory had received since the death of its illustrious Author.

The explanation was as strict and complete as one founded on methods of approximation (and such are the methods of Physical Astronomy) could be. The disturbing force in the direction of the tangent was equally a condition of the problem (see p. xlvii.) with that other resolved part which acts in the direction of the radius. If the latter alone had been the condition, then indeed, as it is shewn in the eleventh Section of the *Principia*, on the principles of the ninth, the *Progression* of the apogee, as a general effect, would have ensued. But it might also have ensued, if the law of attraction had been other than that of the inverse square of the distance; if, for instance, it had been that which Clairaut proposed it should be. In this latter case, indeed, by a proper adjustment of the coefficients of the terms (see p. xlix.) the exact quantity of the progression might have been made to result: but the assumed Law of Gravity would not thereby have been proved to be the true one; nor does the exact quantity of the *Progression* of the apogee deduced from the force varying as the inverse square of the distance set at rest the question about the truth of that latter law. But it does much more than the other result. It furnishes an additional argument in favour of that law which solves so many other phenomena: and, as much as an individual result can do, proves it to be true.

But, as the fact was, and indeed as it was likely to be, it went much farther. The Progression of the apogee having been, for some time, (after much zealous discussion, the most able mathematicians concurring), deemed a fact against Newton's Theory, and then being proved to be on its side, seemed to make that theory quite secure. The System of Gravity by being shaken settled more firmly. It apparently derived strength from the very embarrassment from which it had been extricated.

Clairaut and his contemporaries treated of the variations of some of the other elements of the Lunar orbit; of the inclination of its plane, and of the place of its node: but, nearly, after Newton's manner. Indeed it is difficult to add either to the perspicuity or accuracy with which those subjects are treated of in the eleventh Section and third Book (Propositions 30, 31, &c.) of the *Principia.* Euler, however, in his theory of the Moon altogether abandoned Newton's method: which it is essential to remark, not merely because the former differed from the latter, but because it contains the germ of a general method of determining the variations of the elements of a planet's orbit (see p. 406. of this Work).

The great Author of the *Principia*, as it is plain from various parts of that Work, considered the phenomena of the planetary motions to confirm his theory: but he viewed them as confirming it chiefly by their observance of the *elliptical laws*. The transgressions of those laws, or the inequalities arising from perturbation, although, under different circumstances, as certain a consequence of the principle of attraction as the laws themselves, would not well serve to establish it. Newton considered them to be too minute: indeed, in one part of his Work, he assumes the quiescence * of the aphelia as a phenomenon for establishing the law of the inverse square of the distance. But Clairaut, after discussing the Lunar theory, perceiving that the differential equations and their solutions, would, with slight alteration, apply to the case of the Earth disturbed by Mars, or Jupiter, so applied them. Dalembert, in the second volume of his *Opuscules* did the same: so that, if the determination of the exact quantity of the progression of the Lunar apogee be reckoned the first great addition made to Newton's system, that of the planetary inequalities was the second.

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* Accuratissime autem demonstratur hæc pars propositionis per quietem apheliorum.

The methods (and peculiar ones are required) by which Clairaut deduced the planetary inequalities, are to be found in a Memoir published by the *Academy of Sciences of Paris* in the volume of their Acts for the year 1754. This Memoir of Clairaut's is eminently perspicuous and fertile in invention: it contains the principles of the various analytical contrivances (as they may be called) by which the difficulties that occur in the planetary theories may be overcome: and, on that account, it serves as an useful introduction, and, indeed, commentary to the more elaborate Treatises of his successors.

The two Memoirs of Clairaut, that of 1745 and of 1754, contain the Lunar and Planetary Theories. But, besides these, their Author presented in 1750 the substance of the former, under the form of a Memoir, to the Imperial Academy of Russia. The Academy had proposed a Prize on the subject of the Lunar Theory: and Clairaut's solution of the proposed question * was published at St. Petersburg in 1752. A second edition of the same Work was published at Paris in 1765.

Nearly about the same time Clairaut's contemporaries, Dalembert, Euler, Thomas Simpson, and Mayer published their Researches on the Lunar and Planetary Theories. Dalembert in the first and second volumes of his *Recherches* * and in several Memoirs inserted in the volumes of the Academy of Sciences at Paris: Euler in the *Petersburg Acts* and in two separate Treatises on the Lunar Theory: Mayer in an elaborate Work entitled *Theoria Lunæ* from which the Lunar Tables, known by the name of Mason's, were constructed: and Thomas Simpson in a volume of Tracts published in 1754.

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* The question to be solved was, 'An omnes Inæqualitates quæ in motu Lunæ observantur, Theoriæ Newtonianæ sint consentaneæ, et quænam sit vera Theoria omnium harum Inæqualitatum, unde locus Lunæ ad quodvis tempus quam exactissime possit definiri?'

The researches of the two latter mathematicians are principally confined to the Lunar theory: those of Mayer are most abstruse: well adapted indeed for the construction of Tables, but not at all for the convenience of the Student: they are presented under a most repulsive form. But it is not so with Simpson's Essay: planned with consummate mathematical skill †, it possesses, besides, considerable perspicuity.

The three other mathematicians, Clairaut, Dalembert and Euler may be called the Authors, under Newton, of the *Planetary Theory:* a theory which assigns not solely the *elliptical* places of planets, but the *inequalities* of those places caused by mutual perturbation; and which, besides, as an ulterior object assigns the changes, caused by that same perturbation, in the positions and dimensions of the orbits of planets. Newton, as we may collect from several passages in his Work, knew that these inequalities of the places of planets and of the *elements* of their orbits, must, on his theory, subsist; but, he has no where particularly considered them: either judging them to be inconsiderable, or to involve no peculiar difficulty. Indeed, from a passage in his *Principia* we may presume, with considerable confidence, that he did not suppose the theory of Jupiter and Saturn to be under the latter predicament. The fact, however, is that it does present peculiar difficulties: and so thought the Academy of Sciences at Paris; since, about twenty years after the death of Newton, it proposed, as the subject of its prize, the Theory of Jupiter and Saturn. Euler was a competitor for that prize, but, on investigating the subject, experienced such difficulties that he judged them to be greater even than those which the Lunar Theory presented, 'Car pour peu qu'on s'enfonce dans cette recherche on s'appercevra bientôt qu'elle est beaucoup plus difficile que celle du movement de la Lune, qu'on a jugeé pourtant jusqu'ici la plus difficile recherche de l'Astronomie,' (*Prix de l'Academie des Sciences*, tom. VI. 1748.)

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* Recherches sur differens points dans le système du Monde.

† The *Tracts* of Thomas Simpson were published in 1754, and its Author, in his own way, without (it would so seem) any help from his countrymen, or communication with foreigners, deduced the several Lunar equations, and, rightly, (see Chap. XIII. of this Work) the *progression* of the Lunar apogee. With better opportunities he would have been, at the least, not inferior to any of the first *set* (as we have called them) of Newton's successors. But Clairaut and Dalembert had several advantages over him: they were distinguished members of a learned Academy, in continual intercourse with men of Science, ambitious, emulous of each other, and patronized, on account of their abilities, by the great. There was very little, if we may rely on his biographer, to stimulate or aid the efforts of our countryman. From an obscure station he was transferred to a laborious occupation, with little leisure, and that melancholic or made less by the influence of bad habits.

It is not easy on subjects such as Euler is speaking of, to assign their degrees of difficulty. Jupiter revolving round the Sun and disturbed by Saturn, is a case similar, in its general character, to the Moon revolving round the Earth and disturbed by the Sun. Each case requires the same differential equations and similar methods of approximation. To a certain extent the conduct of both processes of solution is the same. But each, when nearly examined, has its peculiar difficulties: and, indeed, although by the import of terms, a *general* solution of the Problem of the Three Bodies might suit all cases: whether Venus in her orbit were disturbed by Jupiter or the Earth; yet the fact is otherwise: and the principal merit, which we have said to belong to Clairaut for his Planetary Theory, consists in having so adapted or modified his general formulæ as to suit each particular case (see Chapters XVII, XVIII, of this Work).

The difficulty which was met with in the theory of Jupiter and Saturn was not like any that had occurred either in the Lunar or in any other *planetary theory* [*]. It was indeed peculiar and of this kind. The approximate solution of the differential equation (that by which Clairaut and his contemporaries had expressed the conditions of the Problem of Three Bodies) gave, as in the case of the Moon and of the planets, terms expounding certain inequalities : but all such terms involved either the sine or the cosine of an angle ; and, consequently expounded *periodical* inequalities. $A \,.\, \sin.\, n t$, would, for instance, represent one of those terms: now such a term would be nothing at the commencement of any epoch, when $t = 0$; but, having passed through successively increasing values, its maximum, and successively decreasing values, it would again become nothing, when $n t$, by the augmentation of t, should equal 180°: and, that term being passed, $A \,.\, \sin.\, n t$ would become negative, and so continue till $n t$ should equal 360°. The inequality, therefore, expressed by such a term, and consequently, all inequalities affecting Saturn's motion and caused by Jupiter's perturbation (since $A \,.\, \sin.\, n t$ by representing any inequality represents all) would be *periodical*. Saturn's motion, therefore, according to theory was subject to no *secular* inequality: that is, to an inequality which, admitting no alteration of increase and decrease, would either perpetually accelerate or perpetually retard his motion.

But such an inequality it was desirable to find in order to reconcile theory and observation: for, according to the latter, Saturn's mean motion was retarded : which was thus ascertained. The mean motion of a planet [†] is determined by dividing the difference of two longitudes (at each of which the planet was nearly in the same place of its orbit) by the time elapsed. Saturn's mean motion so determined by comparing two modern observations (those made since the revival of Astronomical Science) was found not to agree with the mean motion determined by comparing one of the modern observations with an antient observation. The quantity representing the mean motion was greater in the latter than in the former instance: and like comparisons of other observations established the same fact: the fact of a *retardation* of Saturn's mean motion.

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[*] The *Theory of Venus and the Earth* means that of their respective inequalities arising from their mutual perturbation. The Theories of Jupiter and Saturn mutually disturbing each other, of Mars and Jupiter, bear like significations.

[†] See *Astronomy*, Chap. XXV.

The term *retardation*, as it was used at the time when the question, we are speaking of, was first agitated, was meant to be similar in kind, though opposite in effect, to the term *acceleration* used in Galileo's Theory of falling bodies. In that theory it designated the effect produced on a moving body, by the continued agency of a constant force. It was mathematically expounded by a term such as At^2, t being the time, and A an invariable quantity. The *retardation* of Saturn's motion bore a similar meaning and was similarly expressed: and, in the constructions of Saturn's Tables, it was accounted for, or corrected by a *secular equation* of the form At^2.

Euler, as we have already stated, was unable to trace the cause of such a *secular* equation. There existed then a planetary phenomenon (for such we may call Saturn's retardation) unexplained by Newton's Theory. The theory was not therefore false, but was, at the least, less firm by wanting the support of the explication of so remarkable a phenomenon.

There is indeed on general views, no absurdity, in supposing a secular equation to exist, or that Saturn's mean motion, continuing entire in the elliptical system, should be impaired by Jupiter's disturbing force. Newton himself contemplated the existence of such a circumstance: for, in one of his Works * he speaks of the Universe as about to require, at some time or other, the repairing hand of its Author.

But although the retardation of a mean motion involved no absurdity, yet, whilst it did not appear as a result from theory, it in some degree, however small, impugned that theory. A stronger necessity, however, for explaining the retardation arose soon after the unsuccessful termination of Euler's researches. For, certain theorems were deduced from theory which, were that true, proved the impossibility of a retardation of a mean motion. A secular inequality was therefore not only not made out, but was shewn to be incompatible with other results derived from the principle and Law of Gravitation. Of these results and their Authors we must now speak.

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* 'Some inconsiderable irregularities excepted, which may have risen from the mutual actions of comets and planets upon one another, and which will be apt to increase till this system wants a reformation.' *Optics*, Query 31.

The Authors were Lagrange and Laplace, who belong to the second set of Newton's successors: but amongst the whole list of those successors there are no brighter names: whether we consider their accurate and extensive researches, or their inventions and discoveries. The former of these mathematicians resumed, at first with imperfect success, the attempt in which Euler had been foiled. Then Laplace, having, in his first Essays, obtained an expression for the secular equation of the mean motions of the planets, applied it to the case of Jupiter and Saturn, and found that it became nothing. In other words, their mean motions, abstracting periodical inequalities, were invariable. This is the result of which we have just spoken, as being at complete variance with the fact of a retardation of Saturn's motion. Soon after this result of Laplace's, Lagrange resuming his investigations, obtained a similar one and under a better form. In the *Memoirs of Berlin* for the year 1776, he appears as the Author of that remarkable formula *, from which the invariability of the mean distances of planets may be inferred.

Laplace, on the same subject, obtained another result with the same bearing as the preceding. It was this; the sum of the masses of the planets divided respectively by their mean distances, is, when account is made solely of those inequalities that have very long periods, nearly a constant quantity. If therefore the major axis of Saturn, and, consequently, its period, be increased, the major axis of Jupiter, and, consequently, its period would be diminished. The *retardation*, therefore, of Saturn's motion ought to be contemporaneous and concomitant with Jupiter's *acceleration*. And this latter fact (which Halley had noted) was established by a comparison of observations like that which had been used in proving the former one. This theorem, then, of Laplace's contracted the enquiry; it proved that something like a *secular* retardation might take place: for of such description would be an inequality of a *long period* diminishing the planet's longitude: it also directed the enquiry; since, if explanation were to be had, it shewed that it must be sought for amongst inequalities of a long period.

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* See Chapter XXI. of this Work.

Amongst such inequalities, Laplace, after long research, detected the causes of Saturn's *retardation* and of Jupiter's *acceleration*. It had been usual, in constructing the differential equation which expressed the conditions of the disturbed body, to reject the terms that involved the cubes, the fourth powers, &c. of the eccentricities, because such terms, the eccentricities being minute, became very minute. But amongst the terms so rejected, and involving the cubes of the eccentricities, there were certain terms under peculiar circumstances, which were these; the terms, if integrated, would receive very small divisors, and might, for that reason, become of *retainable* magnitude. Laplace found that these terms when integrated, expounded an inequality of a very long period. An inequality of that kind, retarding Saturn's motion during certain and long portions of its period, would appear to act like a *secular* inequality: and we cannot wonder that Saturn's motion was judged to be subject to such an one, when we consider that the period of that *periodical* inequality which Laplace detected and assigned as the true cause of the *retardation* exceeded nine hundred years.

Laplace assigned a similar, or, rather, the same cause to Jupiter's acceleration; and, (which is the only test of a true explanation) shewed that the computed period of the quantities of the *retardation* and *acceleration* agreed with the observed, (see Chap. XIX. of this Work.)

The peculiar and characteristical condition of the preceding inequality is the great length of its period: and in that circumstance we find the reason why it became so blended with the mean motion as not thence to be disengaged by observation alone. Its extrication is due entirely to theory.

The inequality is peculiar to the theory of Jupiter and Saturn: its special cause is to be sought for in the near commensurability of the mean motions of Jupiter and Saturn: which mean motions are nearly as 5 to 2. But we are thus referred rather to the *mathematical* cause than to any simple or palpable explanation of the phenomena. For, certainly, it is not easy to perceive any thing in the circumstance of the near commensurability of the mean motions of two planets which should occasion one, by a slight modification of its disturbing force, to accelerate the other during four hundred and fifty years, and then to retard it for an equal period.

The case is not singular: almost all the *abstruse* results of Physical Astronomy, as well as of any other branch of science, must be in similar predicaments. They are produced by the combined operation of several causes, acting for considerable periods, and under circumstances continually varying. The argument (could it be stated in common language) which should connect the several parts of this series, and so join the principle with the result, would needs be tedious and embarrassing.

On such occasions the symbolical language of the mathematics comes to our aid and shortens, or makes easier, the processes. It conducts the steps either along the Geometrical or the Analytical method: and it is principally in intricate investigations that the superiority of one method above another is shewn *.

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* Take the methods as we now find them, and the superiority of the Analytical above the Geometrical method, for efficiency, or for the obtaining of results, is indisputable. One of the results not to be obtained by the latter is the one just mentioned in the text, namely, the *retardation* of Saturn's mean motion: a second is the *progression* of the Lunar apogee: a third the *acceleration* of the Moon's mean motion: a fourth the invariability of the mean motions of the planets. If the

Another of Laplace's discoveries (and by which his reputation has become, and deservedly, so great) is that of the cause of the *acceleration* of the Moon's motion: which, considered as a fact or phenomenon, is precisely of the same nature as the *acceleration* of Jupiter's motion, and was detected by Halley by a similar comparison of observations*. It depends, however, on a cause totally dissimilar.

This fact of the Moon's *acceleration* being, as a result from theory, an abstruse one, is, in respect of its explanation, under those predicaments which have been just described (see p. lix.). But, if we assume, as established, certain results, the explanation may be made intelligible without the aid of symbolical processes. The result to be assumed is *the diminution of the eccentricity of the Earth's orbit by the disturbing forces of the planets.* On this result the explanation may be thus founded.

The Moon's gravity to the Earth is altered by the Sun's disturbing force. In one synodical revolution, making account of the diminutions and augmentations, and the former prevailing, it suffers a *mean* diminution. But this diminution depends, in respect of its quantity, on the Sun's distance from the Earth. The greater the distance the less the diminution. The diminution then is greatest in the Winter and least in the Summer months. Now an increase of the Moon's period is a consequence of its diminished gravity. There will, therefore, be several increased periods during the year, which must all be taken into account in determining the mean period, and, thence, the *mean* motion. The mean period so determined would be the same, whatever the year, whether the 220th, or the 1780th of our æra, if the mean diminutions of the Moon's gravity remained the same. But the mean diminutions would not be the same if the

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Geometrical method had been adhered to, Newton's system would have been deprived of more than half its supports. The great Author himself was obliged to abandon it, and to have recourse to the other method: witness, amongst many others, the demonstrations by which he determines the variation of the Moon and the motion of the Nodes.

* See p. lv. of this Preface, also Chap. XXXII. of Astronomy.

relation between the Earth's distances from the Sun should be altered, which it would be by a change in the eccentricity of the Earth's orbit. And this change we have assumed to happen from the disturbing forces of the planets.

As long, therefore, as the disturbing forces of the planets by reason of their *configuration* shall continue to diminish the eccentricity of the Earth's orbit, so long will the Moon's mean motion be *accelerated:* but, after a certain period, (a very long one), the disturbing forces will have a contrary effect, and will increase the eccentricity: and then the Moon's motion will be *retarded.* This explanation and the preceding one (see pp. lvii, &c.) are due entirely to Laplace, and are striking instances of the superiority, with regard to efficiency, which the Analytical possesses over the Geometrical method.

In the preceding explanations the *variability* of the eccentricities and *constancy* of the mean distances have been spoken of: and, in fact, it has been assumed that the former will vary from the disturbing forces, and that the latter will remain unchanged although those same forces act. This last result is by far the most curious of the two. When we perceive, by the agencies of the *third* bodies (as they may be called) almost every part of the elliptical system disturbed, the place of the planet in its orbit, the node, inclination, and aphelion, we are almost led to believe that the major axis would not be suffered to remain exempt from change. It is, however, exempt: and the proof of this with the formulæ for the variations of the elements of the orbit of a planet (which indeed include that proof) are nearly the last, but not the least brilliant of the results in Physical Astronomy, which, under the guidance of modern mathematicians, and by the analytical method, have been arrived at. Of these we will now briefly speak.

The elements of a planet's orbit are the place of its node, of its aphelion, its inclination, eccentricity, and major axis. The *variations* of the first (technically called the *Regression* of the node) and of the third were deduced by Newton; the same great Author has treated slightly of the fourth, imperfectly of the second, and not at all of the fifth; he has not declared any opinion whether or not it were subject to change from disturbing forces. The first set of Newton's successors, Clairaut, Dalembert, Euler, Mayer and Thomas Simpson, do not seem to have interested themselves on this subject. The first who led the way to that species of investigation which has since been so successfully prosecuted, is Lagrange. In the *Memoirs of Berlin* for 1776, he deduced a remarkable expression for the variation of the axis major; and, from that time, his own efforts, and those of Laplace, have, on this subject of enquiry, been directed to the finding out of similar* expressions for the variations of the other elements: and not only have similar or symmetrical expressions been found out, but expressions very simple and easy of application.

These formulæ for the variations of the other elements are either new or much altered: but that which expresses the variation of the major axis remains under its original form.

It follows from that formula (as its Author shewed in 1776) that the mean distances of the planets, and, consequently, their mean motions, although subject to *periodical*, are exempt from *secular* inequalities. They receive, if we estimate them by a sufficiently long interval of years, neither acceleration nor retardation. 'Motus Planetarum in Cœlis diutissime conservari posse,' is true, though not in the sense Newton meant it to be. If we put faith in the results of the modern analysis, we may even predict when Saturn, no longer retarded, shall begin to be compensated for his loss of motion.

The planetary system, therefore, if we regard the mean motions, will endure in its present state. In that respect it will be (as it is technically said) *stable*. Were it otherwise, if the mean motions were continued to be changed and the same way, the system would be tending towards a kind of extinction. If, for instance, Saturn should continue to be retarded and Jupiter accelerated, the former (from the relation between the mean motions and distances) would continually recede from the Sun till it almost ceased to acknowledge its influence: the latter would continually approach the Sun till it fell into it.

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* See *Memoirs of Berlin*, 1776, pp. 199, &c. 1781, 1782. *Mecanique Analytique*, edit. 2. 2de Partie, pp. 102, &c. *Acad. des Sciences*, 1784, 1785. *Mec. Cel.* 1me Partie, Liv. II. pp. 344, and Supplement au III^e Volume.

But the planetary system is *stable* with regard to some of the other elements; and the formulæ which we have alluded to (see p. lxii.) have been made to afford results than which there are none more curious in the whole scope of Astronomical science.

It is easy to see that some of the elements of a planet's orbit may vary without *unsettling*, (as we may say) the system, or affecting its *stability:* this must be the case with two of those elements on which the position of an orbit in fixed space depends. The place of the apogee and the place of the nodes of the Earth's orbit, for instance, may be changed without affecting the vicissitudes of seasons or the degrees of light and heat which the Sun is supposed to impart. Provided the greatest distance remains the same, the Sun's reaching that distance, whether in the sign of Leo or in that of Virgo seems to be a circumstance that entails no consequence. But it must be otherwise with changes in the inclination, and eccentricity. If the latter should increase, the Earth (making that planet still the instance of illustration) would in its perihelion arrive at a continually less *least* distance from the Sun till it reached that body, and in its aphelion at a continually greater *greatest* distance till that distance became the whole major axis. And, under such circumstances, the vicissitudes of heat and light, inasmuch as they depend on distance, would be increased. So it might happen with a change in the inclination of a planet's orbit. The Earth, for instance, is made by the disturbing forces of the planets continually to revolve in a different orbit. The *ecliptic* traced out for the year 1750, is different from the ecliptic of the present year. The consequence of which now is, that the inclination of the ecliptic to the equator, or, technically, the *obliquity of the ecliptic* is diminishing: were this diminution to continue, there would, at length, be no distinction, in as far as it depends on the Sun's declination, between Summer and Winter; there would ensue a kind of perpetual Spring. But the facts, if we take as such the results of theory, are different. The plane of a planet's orbit if inclined, by the perturbation of the other planets, *towards* an assumed fixed plane, will not be perpetually so inclined: but, having reached a certain limit of inclination, begins to be more inclined *from* the fixed plane, and to return towards its former state and another limit. Between this last limit and the former it will continue to oscillate. And thus it must happen with the obliquity of the ecliptic. The plane of the Earth's orbit like that of the orbit of any other planet, will not be moved always the same way, but will oscillate: and the obliquity, diminished to a certain extent, will begin to be increased and successively to reassume its former positions.

In like manner the eccentricity of the Earth's orbit, as well as that of any other orbit, will suffer neither perpetual augmentation nor perpetual diminution, but will vibrate between certain and assignable limits.

These results, certainly very curious, depend on these theorems: the first, in which the stability of the system, relatively to the inclinations, consists, is this: *the sum of the masses of the planets multiplied respectively by the squares of their inclinations to a fixed plane and the square roots of their mean distances, is a constant quantity.*

The second theorem of *stability*, relative to the eccentricities and similar to the preceding, is this. *The sum of the masses of the planets multiplied respectively by the squares of the eccentricities of their orbits, and the square roots of their mean distances is a constant quantity**. It is an easy consequence from these theorems, since both the eccentricities and inclinations, at any epoch at which they were known, were very small, that neither before that epoch could they have exceeded certain limits of magnitude, nor can they after. Take, for instance, the Earth's eccentricity. The term which is formed by multiplying its square, by the mass and square root of the mean distance, can, by the above theorem, never exceed that value which it will have on making the eccentricities of the orbits of the other planets nothing. From such a maximum of the term we may compute the value of the eccentricity, which, since the mass and mean distance are invariable, must also be a maximum and limiting value. And similar inferences shew that the eccentricities of the other orbits can never reach, or, at the most, can never exceed certain limits. So that, as it has been before observed, the eccentricity of each orbit *oscillates* about a mean state.

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* See Chapters XXII, XXIII. of this Work.

The theorem of the inclinations, exactly resembling that of the eccentricities, admits of inferences similar to those which have been just deduced. The plane of each orbit perpetually oscillates about a mean state of inclination.

The Earth, then, if we give belief to the above results, will for ever revolve, as it does now, in a nearly circular orbit, at the same distance from the Sun, and having its axis equally inclined to the plane of the ecliptic. It will be subject to periodical but not to secular inequalities; and the same forces which disturb it from its mean state, will, after long periods, repairing what they have undone, restore it to the same.

These results are certainly amongst the most interesting of Physical Astronomy: but (and it may be considered a subject of regret) they are not easily arrived at; as it often happens with beautiful scenes in nature, the approaches to them are very rugged and intricate.

The series of proofs of his Theory begun by Newton is not yet concluded. Indeed it can have, properly, no termination. We have, however, already far advanced beyond that term, which, in one quarter, Kepler thought to be prescribed to Astronomical research*. Better instruments and more numerous observations will, probably, continue to present new phenomena for explanation. We have seen in the preceding pages what has already been done in that way, since the publication of the *Principia:* and, indeed, no farther back than the year 1754, whilst some Astronomers doubted of the diminution of the obliquity of the ecliptic as a fact of observation, M. Monnier denied that it could be a result of theory *.

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* Nullam planetæ relinqui figuram orbitæ præterquam perfecte ellipticam. *De Stellâ Martis,* p. 285.

The diminution of the obliquity, however, is now as well established by observation as by theory; but there are other points, such as the variations of the inclinations of the planet's orbits, &c. in the shewing of which, theory has gone before observation: and which, therefore, ought rather to be viewed as results than as phenomena. We cannot hope to view them in this latter character except after accurate observations made during a long interval. Their annual quantities are much too minute to be discerned. We must wait till we are able to note their accumulated effects.

Till that happens, these small inequalities deduced from Theory cannot be said either to weaken or confirm it: but we may presume they will have the latter effect: since, hitherto, like inequalities, as they have been successively deduced, have invariably been found to agree with observation. The series of proofs in establishing Newton's system has increased and is still increasing; so that the system has at least that circumstance † as a test of its being founded in truth.

But they are the larger phenomena, the Lunar and planetary inequalities, whether of their places in their orbits, or of the variations of the elements of those orbits, that ought chiefly to be referred to for confirming Newton's Theory. They furnish a long succession of powerful proofs. It is, indeed, easy to assert that Descartes's system has given way to Newton's, as, heretofore, one unreal system gave way to another: but then he who makes such assertion, should point out some notable phenomenon

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* 'M. La Monnier nie absolutement, que l'action des planetes pourroit produire un tel effet sur la terre et ce même sentiment paroit-assés general, que suivant la Theoriè de Newton la situation du plan de l'ecliptique ne sauroit être sensibilement altéreé.' *Mem. Berlin*, 1754, p. 229.

† Quæ, enim in natura fundata sunt, crescunt et augentur.' *Novum Organum.*

(such as the progression of the Lunar apogee) within Newton's system but not obeying its laws. Or he must be content to abide the trial of the other test and examine that series of confirmations (the agreement of the computed and observed phenomena) which have just before been spoken of. This second test is, indeed, to him who makes it, most formidable: for, the computations of the quantities and laws of the phenomena are conducted (as most of them must necessarily be) by the most refined and intricate processes of calculation.

There is one method, indeed, of eluding these difficulties. Newton's Theory may be brought to a test by mean of the Lunar Tables, which are partly constructed by it, and which are intended to serve for many years to come. If from these Tables we take the Moon's place, it is rarely found to differ from the observed place by more than fifteen seconds. Before Newton's discoveries, the error between the computed and observed places amounted to six minutes: although the coefficients and arguments of several of the principal *equations* had (from observation alone) been determined by the antient Astronomers and by Kepler and Tycho Brahé. The modern Tables contain many more equations than the antient, and, for that reason, are better. These equations have been deduced, (the forms of their arguments, at least), from the Theory of Gravity: and there will necessarily arise a very strong presumption for its truth, if the Tables so constructed, give year after year, and many years after their construction, the Moon's place to that exactness which has just been specified. This is a kind of test which may be resorted to by a person although he be not deeply versed in mathematical science: and, which, besides, may easily be resorted to by comparing Burg's Lunar Tables, and the Greenwich Observations: the former computed previously to the latter, and the latter not so made as purposely to uphold Newton's system.

The system of Newton is established on the Theory of Gravity; on its principle and law. The parts composing the system are phenomena of the same class, like effects, or results produced, on mechanical principles, from the same cause; the cause being no occult quality but being always similarly expounded by some line or space (as was shewn in pages xxxv, xxxvi.) capable either of being algebraically expressed or arithmetically valued. The mode by which gravity *causes* its effects (such spaces as we have just spoken of) is beside the scope of the Physical Astronomer.

It is nevertheless a circumstance extremely curious that effects, such as are those of gravity, should be produced; that apparently so small a body as Mars, for instance, should be able sometimes to impede, and at other times to expedite the Earth in its course. The more we reflect on this matter the more mysterious it appears. It is truly wonderful that planetary influence should exist, and that the ingenuity of man should have detected it. Astronomy reveals things scarcely inferior, in interest, to the mysteries of Astrology. It does not indeed pretend to shew that the planets act on the fortunes of men, but it explains after what manner and according to what laws they act on each other.

The Author returns his thanks to the Syndics of the University Press for the liberal assistance afforded him in printing the present Volume.

ERRATA IN PREFACE.

P. x. l. 24. for 'distance' read 'space'
P. xi. l. 15. after 'results' dele comma; and line 26, after '*Astronomy*' instead of , put ;
P. xiv. l. 4. instead of a comma after 'processes' place it after 'being'
P. xxiv. l. 3. from bottom, after 'of' place a comma.

A

TREATISE

ON

PHYSICAL ASTRONOMY.

CHAP. I.

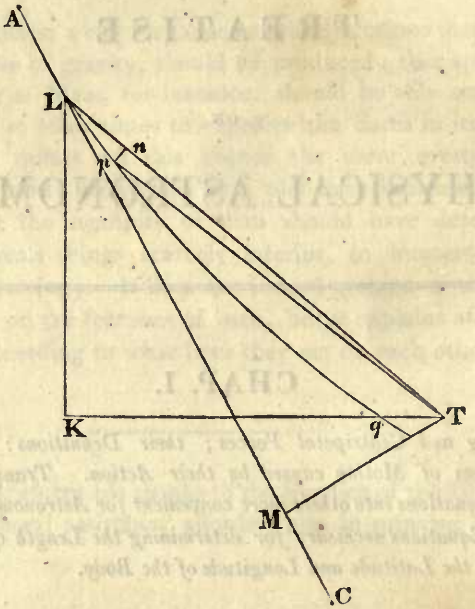
Accelerating and Centripetal Forces; their Definitions: Differential Equations of Motion caused by their Action. Transformation of those Equations into others more convenient for Astronomical purposes. Three Equations necessary for determining the Length of the Radius Vector, the Latitude and Longitude of the Body.

If a body be supposed to be projected from the point A in the direction AC, or if it be merely supposed moving along the line AC with a certain velocity, then, according to the first law of motion, the body, if not compelled to change its state by any *impressed force*, will continue to move uniformly in the same direction AC.

If the body do not continue to move uniformly, or if, during any interval of time, its velocity suffer either increment or decrement, then such change in the uniform motion, or such increment or decrement of velocity, is said to originate from an *accelerating or retarding force*. Again, if the body do not continue to move in the same direction, or, if it be *deflected* or caused to deviate from the line LMC, then such deflection is said to arise from some force, which is variously denominated: it may be centripetal, or repulsive, or disturbing. A force, in fact, is denominated according to the circumstances under which it acts.

For instance, if a body moving in the direction *LC*, be solicited besides by a force *f* according to that same direction, then,

See Same fig. p. 14.



such force *f* produces no deflection from the line *LC*, but solely an acceleration of motion, and accordingly it is called an *Accelerating* force; and, if we consider the point *C* to be a centre towards which the body tends, it may also be called a *Centripetal* force.

If, however, the body moving in the direction of *LC*, be solicited at *L* by a force, acting from *L* towards *T*, then such force produces at once both deflection and acceleration. As a centripetal force, it solicits the body towards *T* as a centre, and deflects it from its right-lined course *LC*. As an accelerating force, it produces an acceleration of motion in the direction of *LC*, not proportional to its whole quantity, but to that part of it which is expounded by $\frac{LM}{LT}$, or cos. TLC. For, if we draw

TM perpendicular to *LC*, and consider *LT* to represent the entire force, *LM* alone produces acceleration in the direction *LC*; since *TM* has no tendency to move the body either from *L* towards *M*, or from *L* towards a similar point in the opposite direction.

The meaſure of an accelerating force is the increment of velocity generated by it during a given time. If the time be increased, the increment will be increased, and in the same proportion. Hence, if f represent the accelerating force generating the increment of the velocity, or, more properly, the differential dv of the velocity, and dt be the corresponding differential of the time, we have, in symbols,

$$dv = f.dt.$$

If v represent the velocity in the direction *LC*, and f be the corresponding force tending from *L* to *C*, and, if V and F be the velocity and force in the direction *LT*, then we have these two equations,

$$dv = f.dt,$$
$$dV = F.dt.$$

Let $TL = p$, and $CL = q$,

$$\text{then } v = \frac{dq}{dt}, \text{ and } V = \frac{dp}{dt};$$

$$\text{consequently, } dv = -\frac{d^2q}{dt},$$

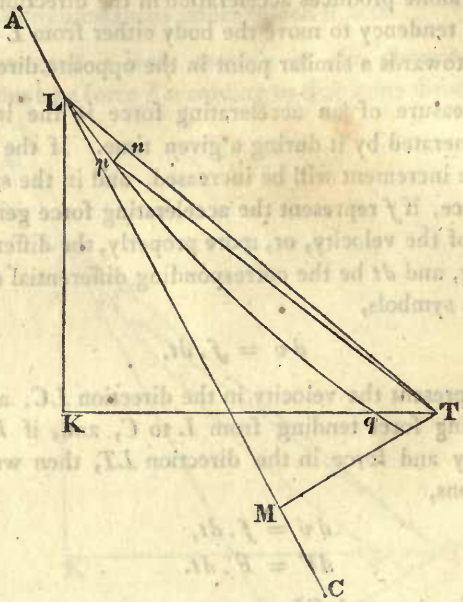
$$\text{and } dV = -\frac{d^2p}{dt},$$

(since, by the action of the accelerating forces by which dv, dV are generated, p, q, are diminished). Hence, instead of the two former equations, we may use these two:

$$\frac{d^2q}{dt^2} + f = 0,$$

$$\frac{d^2p}{dt^2} + F = 0.$$

If the body should be moving in the direction *LC*, and be solicited solely by a centripetal force (F) tending towards *T*, then the force f acting in the direction LC, can only be a resolved part



of the force F, and must equal $F \times \frac{LM}{LT}$; the other part of the resolved force will $= \frac{F \times TM}{LT}$.

Let these two parts of the resolved force be denominated Y' and X', let $TM = x'$, and $ML = y'$, then, in order to determine the body's motion, we shall have two equations similar to the two preceding, namely,

$$\frac{d^2x'}{dt^2} + X' = 0,$$

$$\frac{d^2y'}{dt^2} + Y' = 0.$$

Two lines as TM, ML, perpendicular to each other and serving to determine the place of the body L, are technically called, *Rectangular Co-ordinates* of the body L. Instead of them, we may use two others, such as TK, KL (x, y) and obtain two equations similar to the preceding. For, X' can be resolved into two other forces in the directions of TK, KL, and Y' also may be resolved into two forces in the same directions. Hence, if X, Y, represent the resulting forces in the directions of x and y, we have

$$\frac{d^2 x}{dt^2} + X = 0,$$

$$\frac{d^2 y}{dt^2} + Y = 0.$$

As any force tending from L towards T may be resolved into two forces X, Y, acting in directions parallel to x, y, so may any other force F', and any number of forces F'', F''', &c. be resolved entirely into partial forces acting in the same directions. Hence, if X, instead of representing the resolved part of a single force F, should represent the result of the several resolved parts of the forces, F, F', F'', &c. in a direction parallel to x, and Y the result of the forces parallel to y, the two preceding equations

$$\frac{d^2 x}{dt^2} + X = 0,$$

$$\frac{d^2 y}{dt^2} + Y = 0.$$

would still be true.

We may give to these equations a different form, by substituting instead of X and Y, their values expressed in terms of the entire forces, F, F', &c. For instance, suppose the body to be solicited by a single force F, then, if $r = \sqrt{(x^2 + y^2)}$,

$$X = F \cdot \frac{x}{r}, \quad Y = F \cdot \frac{y}{r};$$

or, since, according to the differential notation, (See *Analyt. Calc.* p. 79.)

$$\frac{dr}{dx} = \frac{x}{r}, \text{ and } \frac{dr}{dy} = \frac{y}{r},$$

(designating by $\frac{dr}{dx}$, $\frac{dr}{dy}$, the partial differential coefficients of $r = \sqrt{x^2 + y^2}$),

$$X = F \cdot \frac{dr}{dx}, \quad Y = F \cdot \frac{dr}{dy};$$

consequently,

$$\frac{d^2 x}{dt^2} + F \cdot \frac{x}{r} = 0, \text{ or, } \frac{d^2 x}{dt^2} + F \cdot \frac{dr}{dx} = 0,$$

$$\frac{d^2 y}{dt^2} + F \cdot \frac{y}{r} = 0, \text{ or, } \frac{d^2 y}{dt^2} + F \cdot \frac{dr}{dy} = 0.$$

F, in the above equations, represents the centripetal force tending towards T. The equations, it is plain, cannot be solved except we assign to F a specific value; and such value will depend on what is called the *Law of the Variation* of the force. Suppose the law to be that of the inverse square of the distance of the body L from T; then, assuming μ to be a determinate quantity, we may represent F by $\frac{\mu}{r^2}$, and the two preceding equations will become

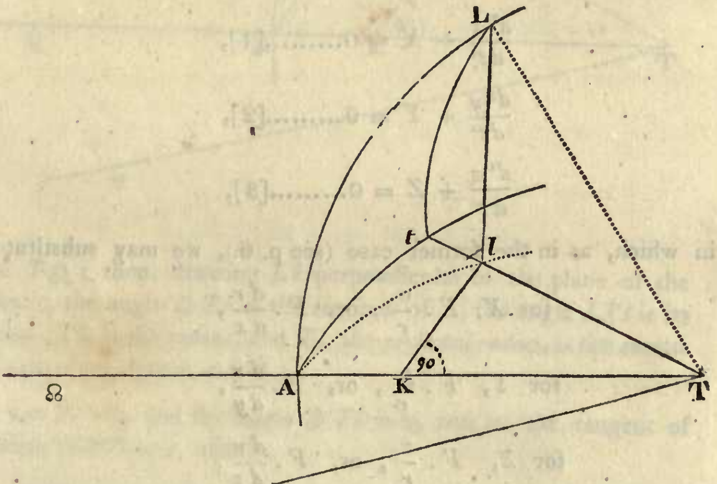
$$\frac{d^2 x}{dt^2} + \frac{\mu x}{r^3} = 0,$$

$$\frac{d^2 y}{dt^2} + \frac{\mu y}{r^3} = 0.$$

Hitherto we have taken account of only one plane, namely, that in which the co-ordinates x, y are situated, and in which the forces X, Y, act. And, it is not essentially necessary to consider more than one plane, so long as the forces, whatever they be, continue to act in it. It might, however, even in these circumstances, be *convenient*, to consider the body's position and motion relatively to a second plane. For instance, the forces that act on Mercury lie almost entirely in the plane of the orbit of that planet. If they did so exactly, still we should find it convenient to introduce, inclined to the orbit's plane, a plane like that of the ecliptic, to which Astronomers are accustomed to refer the positions and motions of heavenly bodies.

But if the forces do not lie all in the same plane, then it becomes absolutely necessary to take account of more planes than one.

As two rectangular co-ordinates x, y, are sufficient to determine the distance of the body L from T in the plane of LK, KT, (see Fig. p. 4.) so, three x, y, z, (z being drawn perpendicular to the plane of x, and y) will determine the distance when the body is either above, or below, the plane in which x and y are. And, whatever be their directions, the forces that act on a body, or on a system of bodies, may all be resolved into three directions respectively parallel to the co-ordinates x, y, z. For, let TK be x, Kl, perpendicular to it, y, and let Ll perpendicular, at the point l, to the plane passing through x, y, be z; then, if the force



acting on L were represented by any line drawn obliquely to the plane of x and y, it might be resolved into two others, one parallel to Ll (z), the other lying in the plane of x and y: and this may be effected by merely drawing from one extremity of the line representing the force, a line parallel to z, and, from the other extremity, a line parallel to the plane of x and y, and produced to the former line. The force represented by the second line, lying either in the plane of x and y, or in a parallel plane, being then, by a second process of resolution, resolved into two directions parallel to x and y respectively, the whole force would be resolved into three others parallel to x, y and z.

In the figure, it is plain that $Tl = \sqrt{(x^2 + y^2)}$, and $TL = \sqrt{(x^2 + y^2 + z^2)}$. Tl, if TL be called the radius, is said to be the *projected* radius.

In order therefore to determine the body's motion, &c. when the forces do not lie in the same plane, we must introduce a third equation similar to the two that have been already introduced. The three differential equations of motion will then be

$$\frac{d^2 x}{dt^2} + X = 0 \ldots\ldots\ldots [1],$$

$$\frac{d^2 y}{dt^2} + Y = 0 \ldots\ldots\ldots [2],$$

$$\frac{d^2 z}{dt^2} + Z = 0 \ldots\ldots\ldots [3],$$

in which, as in the former case (see p. 6.), we may substitute

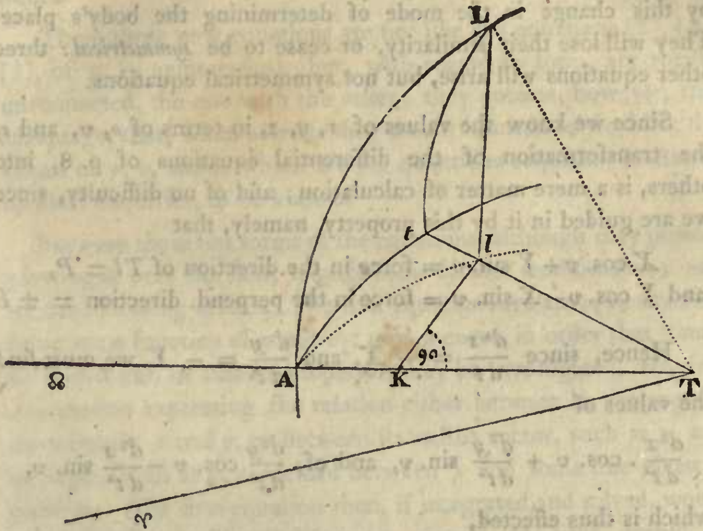
$$\text{for } X,\ F \cdot \frac{x}{r}\ , \text{ or, } F \cdot \frac{dr}{dx},$$

$$\text{for } Y,\ F \cdot \frac{y}{r}\ , \text{ or, } F \cdot \frac{dr}{dy},$$

$$\text{for } Z,\ F \cdot \frac{z}{r}\ , \text{ or, } F \cdot \frac{dr}{dz}.$$

The body's place has been supposed to be determined by means of three rectangular co-ordinates x, y, z; and, there is no other more simple way of determining it. But, if we look to the custom of Astronomers, this is not the usual mode of determining it. A body's place (see *Astronomy*, p. 252.) is made to depend on the length of the radius vector (or on that of the *curtate distance* or projected radius) and on its latitude and longitude. It is made therefore to depend on, one line and two angles, instead of, three lines.

Suppose the plane of $x, y,$ to represent the plane of the ecliptic, and the longitudes of bodies to be measured from the



line T ☊ ; then, drawing Ll perpendicular to the plane of the ecliptic, the angle ☊ Tl is the *longitude* of L, the angle LTl is its *latitude*, TL is the radius, and Tl, the *projected radius*, is the *curtate distance*, (see *Astron.* p. 273.).

Let $Tl = \rho$ and the angle ☊ $Tl = v$, and let the tangent of latitude (LTl) be s, then

$$x = \rho \,.\, \text{cos.}\, v, \text{ or, } = \frac{r}{\sqrt{(1 + s^2)}} \,.\, \text{cos.}\, v,$$

$$y = \rho \,.\, \text{sin.}\, v, \text{ or, } = \frac{r}{\sqrt{(1 + s^2)}} \,.\, \text{sin.}\, v,$$

$$z = \rho\, s.$$

and, accordingly, instead of determining the body's place by $x, y,$ and z, we may determine it, by ρ, v, and s, or by r, v and s.

This change in the conditions of the body's place, will render a change necessary in the mode of expressing the forces. The forces X, Y, acting in the directions of x and y, must be resolved into others acting in the direction of ρ, and in a direction perpendicular to ρ.

The three differential equations of p. 8, will also be affected by this change in the mode of determining the body's place. They will lose their similarity, or cease to be *symmetrical:* three other equations will arise, but not symmetrical equations.

Since we know the values of x, y, z, in terms of ρ, v, and s, the transformation of the differential equations of p. 8, into others, is a mere matter of calculation; and of no difficulty, since we are guided in it by this property, namely, that

$X \cos. v + Y \sin. v =$ force in the direction of $Tl = P$,
and $Y \cos. v - X \sin. v =$ force in the perpend. direction $= \pm T$.

Hence, since $\frac{d^2 x}{dt^2} = - X$, and $\frac{d^2 y}{dt^2} = - Y$, we must find the values of

$$\frac{d^2 x}{dt^2} . \cos. v + \frac{d^2 y}{dt^2} \sin. v, \text{ and of, } \frac{d^2 y}{dt^2} \cos. v - \frac{d^2 x}{dt^2} \sin. v,$$

which is thus effected,

$$x = \rho . \cos. v, \quad y = \rho . \sin. v;$$
$$\therefore dx = d\rho . \cos. v - dv . \rho . \sin. v,$$
$$dy = d\rho . \sin. v + dv . \rho . \cos. v,$$
$$d^2x = d^2\rho . \cos. v - 2dv . d\rho . \sin. v - \rho . dv^2 . \cos. v - \rho . d^2v . \sin. v,$$
$$d^2y = d^2\rho . \sin. v + 2dv . d\rho . \cos. v - \rho . dv^2 . \sin. v + \rho . d^2v . \cos. v;$$
$$\therefore d^2x . \cos. v + d^2y . \sin. v = d^2\rho - \rho . dv^2,$$
$$d^2y . \cos. v - d^2x . \sin. v = 2dv . d\rho + \rho . d^2v.$$

Hence, writing S instead of Z, and ρs instead of z in the last equation, we have these three new equations,

$$d^2\rho - \rho dv^2 + P . dt^2 = 0 \ldots\ldots\ldots\ldots [4]^*,$$
$$2dvd\rho + \rho d^2v \pm T . dt^2 = 0 \ldots\ldots\ldots\ldots [5],$$
$$d^2(\rho s) + S . dt^2 = 0 \ldots\ldots\ldots\ldots [6],$$

in which P, T, and S, represent the results of any number of forces that act upon the body L; the first in directions parallel to ρ, the second in directions perpendicular to the former, and the third in directions perpendicular to the plane of the two first forces.

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* These equations are the same as those which Dalembert has inserted in the 6th Volume of his *Opuscules.*

These three new equations are not like the equations [1], [2], [3], of p. 8, symmetrical, but, with regard to form, are totally unconnected, the one with the other; they possess, however, this advantage, that, when solved, they would immediately exhibit the values of ρ, v, and s, which are the quantities requisite, in Astronomical enquiries, to be known.

But even these last forms of the equations, although they possess advantages over the first, require some farther modification; and for the following reason. The first equation involves ρ, v, and t (P being some function of ρ and v): now a curve, in order that it may be traced out, or that its properties may be investigated, requires an equation expressing the relation either between its rectangular co-ordinates x and y, or between its radius vector, such as ρ, and an angle, such as v, contained between ρ and some line given in position. The first equation then, if integrated and solved, would not define the nature of the curve, since t, the time, would be involved in it: t, therefore, must be eliminated; and its elimination will be the object of a succeeding transformation.

But, as we shall hereafter see, the process will not terminate with that transformation. There will remain to be made another step; a very short one, indeed, consisting merely in substituting, instead of ρ and its functions, $\frac{1}{u}$ and its corresponding functions.

This substitution, beyond what could be presumed from any antecedent reasons, is eminently useful in abridging the process of calculation: and was, probably, rather happily hit on, than found by any scientific clue *.

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* In tracing the connection of the successive transformations, it will be thought, perhaps, that we have rather made a way than found one. It is, indeed, almost necessary, and certainly it is very commodious, to establish, between methods so difficult as those of Physical Astronomy, an

If we were immediately to press forward to those most commodious and perfect forms, which the ingenuity and labour of Mathematicians have given to the Differential Equations of Motion, we should conduct the Student, in the out-set of his career, over too extended a field of apparently barren speculation. It is better to stop for a while and endeavour to collect some useful results. And this we shall be enabled to do by investing the preceding equations with the conditions that obtain in nature, that is, by substituting, instead of the general symbols P, T, and S, the expressions of those forces by which bodies in the Planetary system are mutually drawn towards each other.

The deduction of these results will be the object of the succeeding Chapters; but, in the one that immediately follows, the result obtained is altogether independent of the Law of Centripetal Forces.

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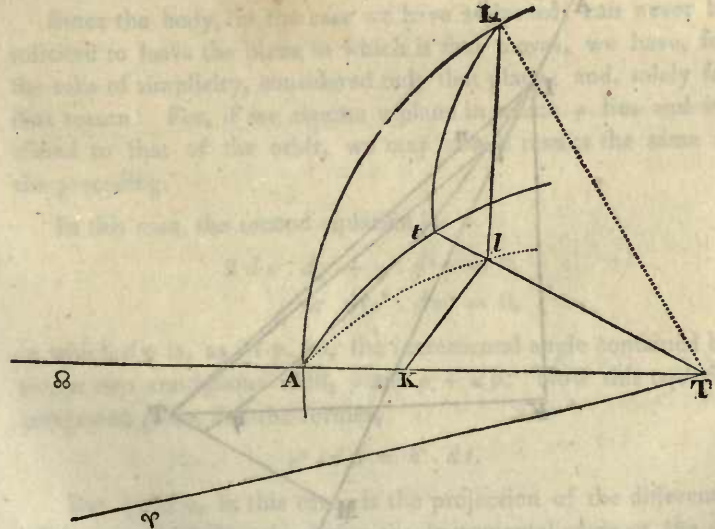
an artificial connection, when no natural one exists. And scarcely any natural one exists. The *approved* methods of science are very different, in their form, from those which their inventors first exhibited, and still more different, probably, from those which were first investigated. Their present compactness and neatness is the fruit of numerous trials and *experiments,* of which the traces are not preserved.

CHAP. II.

Consequences that follow from the Differential Equations of Motion when the Forces acting on a Body in motion are Centripetal, or are directed to one Point only: Kepler's Law of the Equable Description of Areas demonstrated. Variation of the Velocity. The Equable Description of Areas necessarily disturbed, when the Body is acted on by Forces, some of which are not directed to the same Point or Centre.

In the equations (4), (5), (6), of p. 10, the sole condition regulating the forces P, T, S is, that they should act on the body L in directions respectively parallel to Tl, perpendicular to ~~Lt~~; Tl and parallel to Ll. Let us now suppose the forces which act on L to act (previously to any *resolution* of them) solely in the direction of LT. In this case there can be no force to draw the body out of the plane in which it once moves. It is not therefore *essential* (see p. 6,) to consider any other plane than that of the



body's orbit. We may suppose, then, the latter plane and that in which Tl, T☊ are to be coincident; in which supposition, s and $S = 0$, and $\rho = r$, and T, also, $= 0$; for, the sole force (P) acting in the direction LT admits of no resolution in a direction perpendicular to LT. The equation (6), then, of p. 10, disappears, and the equations (4), (5), are reduced to these forms,

$$d^2 r - r\,.\,dv^2 + P\,dt^2 = 0,$$
$$2\,dv\,dr + r\,d^2 v = 0.$$

The second of these equations may be put under this form,

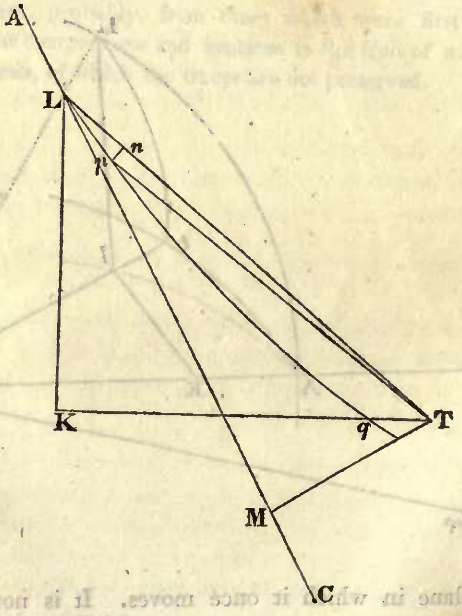
$$2r\,dr\,.\,dv + r^2 d^2 v, \text{ or, } d(r^2 dv) = 0.$$

But, if $d(r^2 dv) = 0$, we have, by integration,

$$r^2 dv = h\,dt,$$

dt being the differential of the time, which is supposed constant, and h being an arbitrary quantity to be determined according to the conditions of any specific case.

The result just obtained is a very remarkable one; it amounts to what is usually known by the name of *Kepler's Law of the Equable Description of Areas.* For, if the body be describing the



curve Lpq, and Tp be supposed to be indefinitely near to TL, (pn being perpendicular to LT), the incremental area

$$LTp = \frac{pn \times LT}{2} = \frac{r\,dv \times LT}{2} = \frac{r\,dv \times r}{2} = \frac{r^2 . dv}{2};$$

and, hence, the differential of the area will be proportional to $h : dt$ and dt, and, accordingly, the integral or the whole area (LTq for instance), will be proportional to the time of the body's describing the arc Lq.

This law of the equable description of areas, Kepler, by observation, ascertained * to be true in the orbits of the planets; and Newton, in the first Proposition of the second Section of the *Principia*, shewed that it was a necessary consequence of the action of a *centripetal* force on a body moving obliquely to a line joining it and the centre of force.

The law, indeed, depends on the condition of the force, or forces, being centripetal: that is, it requires they should act in a line joining the body, and the point which is considered as the centre of the body's motion. In other respects there is no limitation; the force may be of any quantity, and may vary according to any law.

Since the body, in the case we have supposed, can never be solicited to leave the plane in which it first moves, we have, for the sake of simplicity, considered only that plane; and, solely for that reason. For, if we assume a plane in which ρ lies and inclined to that of the orbit, we may obtain results the same as the preceding.

In this case, the second equation is

$$2\,dv . d\rho + \rho . d^2 v = 0,$$
$$\text{or } d(\rho^2 . dv) = 0,$$

in which dv is, as in p. 10, the incremental angle contained between two contiguous radii, ρ and $\rho + d\rho$. Now this equation integrated gives, like the former,

$$\rho^2 . dv = h' . dt.$$

But $\frac{1}{2}\rho^2\,dv$, in this case, is the projection of the differential of the area ($\frac{1}{2} r^2 . dw$, dw being the incremental angle or the differential angle between two contiguous radii r and $r+dr$): which also is constant when dt is; consequently, the area of the *projected* curve varies as the time: and from this result the former (see p. 14, l. 12,) might have been deduced; since, by the principles of projection, the area really described ($\frac{1}{2}\int r^2\,dw$), is to the projection of the area ($\frac{1}{2}\int \rho^2\,dv$) in the ratio of radius to the cosine of the inclination of the two planes: which, since the inclination of the plane of the orbit can never alter, must be a constant ratio.

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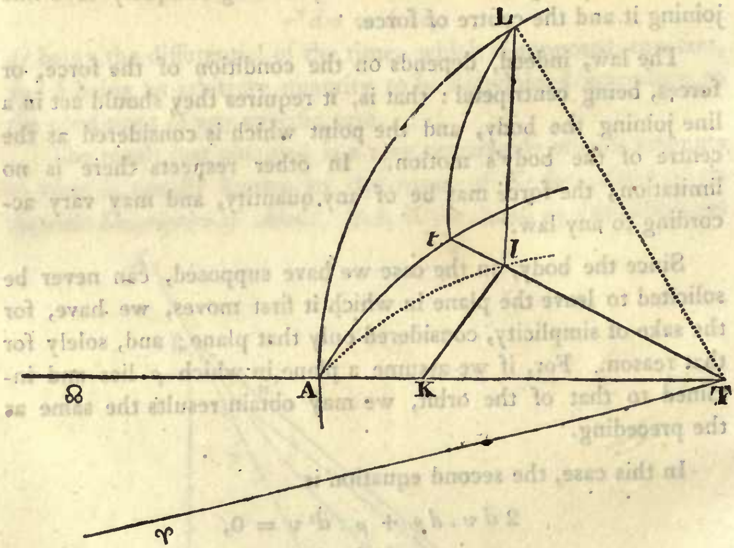
* *Astronomy*, p. 188.

The preceding result may, with little difficulty, be more formally deduced: thus, if ϕ denote the inclination of the planes



LAT, *tAT**, that is, if it equal (see *Trig.* ed. 2. p. 119.) the spherical angle *LAt*, we have, by Naper's Rules, (see *Trig.* p. 136.)

$$1 \times \cos. LAt = \tan. At \times \cot. LA, \text{ or,}$$

$$\tan. v = \tan. w \times \cos. \phi;$$

---

* In the Figure of the text, the dotted line Al represents the curve AL projected on the plane of Al, lK: and At represents part of a great circle lying in the last plane.

$\therefore$ (see *Trig.* p. 98.) $\dfrac{d\,v}{\cos.^2 v} = \dfrac{d\,w}{\cos.^2 w} \times \cos.\,\phi,$

and, since $\rho^2 = r^2.\cos.^2.\,L\,t$ ($L\,t$ being a circular arc),

$$\rho^2.\,d\,v = \frac{\cos.^2 v}{\cos.^2 w} \times \cos.^2 L\,t \times r^2.\,d w \times \cos.\,\phi.$$

But by Naper's Rule, see *Trig.* p. 136,

$$\cos.^2 w = \cos.^2 v\,.\,\cos.^2\,.\,L\,t\,;$$

$$\therefore\ \rho^2.\,d\,v = r^2.\,d\,w \times \cos.\,\phi,$$

and, since $\cos.\,\phi$ is constant,

$$\tfrac{1}{2}\textstyle\int \rho^2.\,d\,v = \cos.\,\phi \times \tfrac{1}{2}\textstyle\int r^2\,d\,w.$$

Hence, the relation between the quantities h and h' is thus defined,

$$h' = h\,.\,\cos.\,\phi\,^*.$$

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[*] Kepler's Law of the equable description of areas has been proved from the equations (4), (5), (6); but, it may readily be proved from the original equations [1], [2], [3]: thus

$$\frac{d^2x}{d\,t^2} + X = 0 \ldots\ldots\ldots\ldots [1],$$

$$\frac{d^2y}{d\,t^2} + Y = 0 \ldots\ldots\ldots\ldots [2],$$

$$\frac{d^2z}{d\,t^2} + Z = 0 \ldots\ldots\ldots\ldots [3],$$

multiply [2] by x, and subtract from it [1] multiplied by y; multiply [3] by x, and subtract from [1] multiplied by z, &c. then the following equations will arise

$$\frac{x\,d^2y - y\,d^2x}{d\,t^2},\ \text{or}\ \frac{1}{d\,t}\,.\,d\left(\frac{x\,d\,y - y\,d\,x}{d\,t}\right) = Xy - Yx,$$

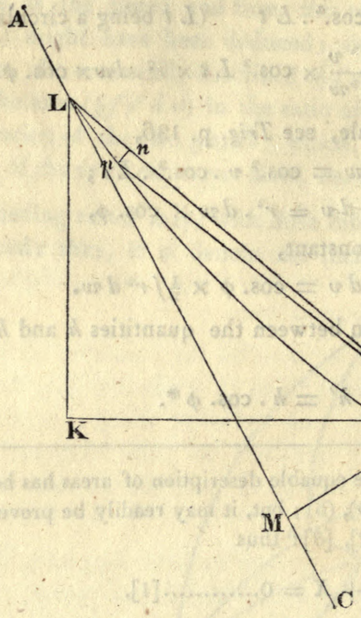
$$\frac{x\,d^2z - z\,d^2x}{d\,t^2},\ \text{or}\ \frac{1}{d\,t}\,.\,d\left(\frac{x\,d\,z - z\,d\,x}{d\,t}\right) = Xz - Zx,$$

$$\frac{y\,d^2z - z\,d^2y}{d\,t^2},\ \text{or}\ \frac{1}{d\,t}\,.\,d\left(\frac{y\,d\,z - z\,d\,y}{d\,t}\right) = Yz - Zy.$$

Now, the sole force (F), being that which acts in the direction of the radius, we have, see p. 8,

$$X = \frac{Fx}{r},\ Y = \frac{Fy}{r},\ Z = \frac{Fz}{r}\,;$$ consequently,

If we call V the velocity with which Lp is described, we shall have



$$Lp = V \,.\, dt\,;$$

$$\text{but } Lp \times \frac{TM}{2} = \frac{pn \times LT}{2} = \frac{r^2 \,.\, dw}{2} = \frac{h \,.\, dt}{2}.$$

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consequently, $Xy - Yx$, $Xz - Zx$, $Yz - Zy$, are all $= 0$;

$$\therefore \text{ we have } d\left(\frac{x\,dy - y\,dx}{dt}\right) = 0,$$

$$d\left(\frac{x\,dz - z\,dx}{dt}\right) = 0,$$

$$d\left(\frac{y\,dz - z\,dy}{dt}\right) = 0.$$

Hence, integrating and assuming c, c', c'', three arbitrary quantities, there result

$$x\,dy - y\,dx = c\,dt,$$
$$x\,dz - z\,dx = c'\,dt,$$
$$y\,dz - z\,dy = c''\,dt\,;$$

and

Hence, $V \times TM \times dt = h . dt$,

$$\text{and } V = \frac{h}{TM},$$

and consequently the velocity is inversely as the perpendicular let fall from the centre of force on a tangent to the curve at the body's place (L). (See Newton, *Principia*, ed. 3. p. 40.)

If the force or forces, whatever they be, do not act in the direction of a line drawn from the body's place to their centre, then the force T is not $= 0$, and the second equation will become

$$d(\rho^2 . dv) \pm \rho . T dt^2 = 0,$$

and, integrating,

$$\rho^2 dv = h . dt \mp dt \int \rho . T dt;$$

consequently, by reason of the last term, the equable description of areas is, in this case, no longer preserved, or, in other words, is *disturbed;* and the force is called a *disturbing* force, because the centripetal force urging L towards T, is imagined to be the proper and natural force, by the action of which alone, the regular and equable description of areas would take place.

Hitherto no mention has been made of the law of the force. In the next Chapter, we will suppose L to be acted on solely by a centripetal force, and that force to vary, as it does in nature, according to the law of the inverse square of the distance between the

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and the halves of the left-hand equations represent, respectively, the differentials of the areas on the planes of x, y, of x, z, and of y, z: or, are the projections of the incremental area ($\frac{1}{2} r^2 dw$) lying in the plane of the orbit; and, by the theory of projections,

$$h^2 = c^2 + c'^2 + c''^2.$$

This process has been inserted in a note, because it is not essential to the result which has been differently deduced in the text; and, it has been inserted partly on account of the importance and the celebrity of its result, and partly as a kind of exercise to the Student, and as a means of shewing how the same conclusions may be obtained either from the fundamental equations of p. 8, or the transformed ones of p. 10.

body or point attracted, and the centre of force or attraction. According to the first condition then, Kepler's Law must, in this case, accurately obtain. And the second condition will conduct us to results equally curious with those that have been already obtained, and to the establishment of two other of *Kepler's Laws* relative to the form of the orbit and the variation of the periodic time.

CHAP. III.

The Centripetal Force is supposed to act inversely as the Square of the Distance. Consequences that flow from it. The Orbit, or the Curve described by the moving Body round the Central, an Ellipse. Kepler's Law of the Squares of the Periodic Times varying as the Cubes of the Major Axes. Kepler's Problem for determining the true from the mean Anomaly. His Law respecting the Periodic Times not exactly true.

THE centripetal force tending towards the point or *centre T* being, by supposition, the sole force that acts on the body, the perpendicular force, which has been designated by *T*, must, for the reasons already assigned in p. 14, be equal nothing; and, since the body can never deviate from that plane in which it once has moved, we may get rid of, or expunge from the calculation, the force *S*, by supposing the plane, to which its action is perpendicular, coincident with the plane of the orbit. If, besides these conditions, we assume μ to be an invariable quantity, and expound the centripetal force P by $\frac{\mu}{r^2}$, (which is to suppose the law of its variation to be according to the inverse square of the distance), the equations (4), (5), of p. 10, will become

$$d^2r - r\,dw^2 + \frac{\mu}{r^2}\,dt^2 = 0,$$

$$2\,dw\,.\,dr + r\,d^2w = 0,$$

ρ becoming in this case r, and v, w.

Now, as we have seen, the second of these equations gives us Kepler's law of the equable description of areas, and the variation of the velocity in terms of the perpendicular. The first, if dt were eliminated, would give r in terms of w and certain constant quantities: it would then give us, what is the object of enquiry, namely, the nature of the curve described. The end to be attained then is very obvious. By means of the two equations in which $d\,t$ (see p. 14,) is constant, we must form another from which $d\,t$ has been eliminated, and containing $d\,w$ as a constant element.

ce p. 226. The conversion of one equation, in which $d\,t$ is constant and $d\,w$ variable, into another in which $d\,w$ should be constant and $d\,t$ variable, is (see Dealtry's *Fluxions*, p. 328. Vince, p. 185. ed. 1. *Prin. Anal. Calc.* p. 90,) a common analytical operation: and so simple, that it may be here inserted without its materially impeding the progress of investigation.

The first equation, employing the general character P instead of $\frac{\mu}{r^2}$, and (since there is, in this case, no necessity for distinction) v instead of w, becomes

$$\frac{1}{d\,t} \cdot d\left(\frac{d\,r}{d\,t}\right) - r \cdot \frac{d\,v^2}{d\,t^2} + P = 0,$$

$$\text{or } \frac{1}{d\,t}\left(\frac{d^2 r}{d\,t} - \frac{d^2 t \,.\, d\,r}{d\,t^2}\right) - r : \frac{d\,v^2}{d\,t^2} + P = 0,$$

and, the second equation integrated (see p. 14,) gives

$$\frac{1}{d\,t} = \frac{h}{r^2 \,.\, d\,v},$$

and the differential of this, supposing (see l. 5,) $d\,t$ variable, and $d\,v$ constant, is

$$-\frac{d^2 t}{d\,t^2} = -\frac{2\,h\,d\,r}{r^3 \,.\, d\,v},$$

which, and the value of $\frac{1}{d\,t^2}$, being substituted in the equation of l. 14, there results

$$\frac{h^2 \,.\, d^2 r}{r^4 \,.\, d\,v^2} - \frac{2\,h^2 \,.\, d\,r^2}{r^5 \,.\, d\,v^2} - \frac{h^2}{r^3} + P = 0,$$

$$\text{or, } \frac{h^2}{r^2 d\,v^2}\left(\frac{d^2 r}{r^2} - \frac{2\,d\,r^2}{r^3}\right) - \frac{h^2}{r^3} + P = 0,$$

now the quantity, within the brackets, is equal $d\left(\frac{d\,r}{r^2}\right)$; therefore, if we make $u = \frac{1}{r}$, and consequently, $-d\,u = \frac{d\,r}{r^2}$, there results

$$-\frac{h^2 u^2}{d v^2} \times d^2 u - h^2 u^3 + P = 0;$$

but $P = \frac{\mu}{r^2} = \mu u^2$; $\therefore$ dividing each term by $-h^2 u^2$,

$$\frac{d^2 u}{d v^2} + u - \frac{\mu}{h^2} = 0,$$

which is an equation between u and v, and which integrated would give us the relation between $u \left(= \frac{1}{r}\right)$, and v, and consequently would determine (see p. 21,) the nature of the curve described. Our attention is therefore naturally directed to the integration of the preceding equation.

If, in the equation $\frac{d^2 u}{d v^2} + u = 0$, we substitute, instead of u, either $a \, . \, \sin. \, v$, or $b \, . \, \cos. \, v$, the resulting equation becomes, as it is technically said, *identically* nothing. Hence, either $u = a \sin. \, v$, or $u = b \cos. \, v$, *satisfies* the differential equation; so must $u = a \sin. v + b \cos. \, v$, and since (see *Prin. Anal. Calc.* pp. 90, &c.) an equation of the second degree requires for its complete integration *two* arbitrary quantities, the last form, viz. $u = a \sin. \, v + b \, . \cos. \, v$, must be the true and complete one: and the condition, either that $a = 0$, or $b = 0$, can only happen in particular cases.

If, in the equation $\frac{d^2 u}{d v^2} + u = 0$, the value of u is $u = a \sin. \, v + b \cos. \, v$, then in the equation, $\frac{d^2 u}{d v^2} + u - \frac{\mu}{h^2} = 0$, the value of u is, $u = a \sin. \, v + b \cos. \, v + \frac{\mu}{h^2}$, in which the arbitrary quantities a and b are to be determined by the conditions of the case.

Now, $\frac{d u}{d v} = a \, . \cos. \, v - b \, . \sin. \, v$; when $v = 0$, the value of $\frac{d u}{d v}$ is a, and when $v = 90^\circ$, the value of $\frac{d u}{d v}$ is $-b$; consequently, between the values of $v = 0$, and $v = 90^\circ$, there is a value of v which renders the corresponding value of $\frac{du}{dv} = 0$: let such value of v be π: then

$$0 = a\cos.\,\pi - b.\sin.\,\pi;$$

consequently,

$$u = b\frac{\sin.\,\pi}{\cos.\,\pi} \times \sin.\,v + b.\cos.\,v + \frac{\mu}{h^2}$$

$$= \frac{b}{\cos.\,\pi}(\sin.\,\pi\,\sin.\,v + \cos.\,\pi.\cos.\,v) + \frac{\mu}{h^2}$$

$$= (\textit{Trig. p. 26.})\ \frac{b}{\cos.\,\pi} \times \cos.\,(v - \pi) + \frac{\mu}{h^2}.$$

In order to determine the quantity b, we may observe that, according to the preceding hypothesis,

$$0 = a.\cos.\,\pi - b\sin.\,\pi,$$

and also, that this equation will equally result from the original one, whether, instead of v, π or $(180^\circ + \pi)$ be substituted, since (*Trig.* p. 9.)

$$a\cos.\,(180^\circ + \pi) - b.\sin.\,(180^\circ + \pi) = -a.\cos.\,\pi + b\sin\,\pi.$$

Hence, $\frac{du}{dv}$ is equal to nothing, both when $v = \pi$, and when $v = 180^\circ + \pi$. Let r' and r'' be the two values of $\frac{1}{u}$ when v is respectively π and $180^\circ + \pi$, then

$$\frac{1}{r'} = \frac{b}{\cos.\,\pi} + \frac{\mu}{h^2},$$

$$\text{and, } \frac{1}{r''} = -\frac{b}{\cos.\,\pi} + \frac{\mu}{h^2};$$

$$\text{whence, } \frac{b}{\cos.\,\pi} = \frac{1}{2}\left(\frac{1}{r'} - \frac{1}{r''}\right),$$

$$\text{and } \frac{\mu}{h^2} = \frac{1}{2}\left(\frac{1}{r'} + \frac{1}{r''}\right);$$

which quantities being substituted in the preceding equation (p. 23, l. 20,) will give the value of u in known quantities.

We may still farther simplify the preceding forms. It has appeared that, if $\frac{du}{dv} = 0$, when $v = \pi$, it also $= 0$, when

$v = (180^\circ + \pi)$. Hence, the two values of $\frac{1}{u}$, r' and r'', corresponding to this value of $\frac{du}{dv} = 0$, are *angularly* distant from each other by 180°: in other words, they lie in the same straight line, and (reckoning from the centre, or their point of junction) towards different parts of it; and since $\frac{du}{dv} = 0$, the extremities of the straight line (which are also those of the distances r', r'') must cut the curve at right angles. Hence, if we make

$$r' + r'' = 2a,$$
$$r'' - r' = 2ae,$$

we shall have

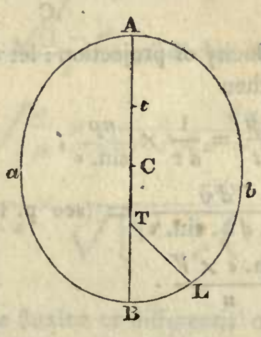
$$\frac{b}{\cos.\pi} = \frac{r'' - r'}{2r''r'} = \frac{ae}{a^2 - a^2e^2} = \frac{e}{a.(1 - e^2)},$$

$$\frac{\mu}{h^2} = \frac{r'' + r'}{2r''r'} = \frac{a}{a^2 - a^2e^2} = \frac{1}{a.(1 - e^2)};$$

and consequently

$$u = \frac{1}{a.(1 - e^2)} \times [1 + e.\cos.(v - \pi)].$$

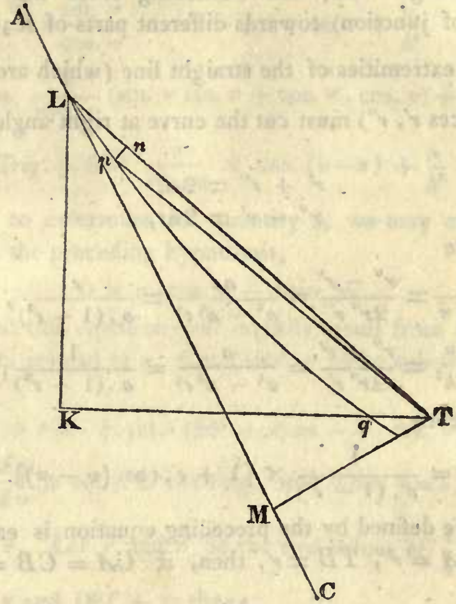
The curve defined by the preceding equation is easily traced out: take $TA = r''$, $TB = r'$, then, if $CA = CB = a$, $TC =$



$\frac{1}{2}(r'' - r') = ae$, and (see 1. 7.) the curve at A and B is perpendicular to the line AB, and must be of an oval form, the part AaB being similar to AbB. The points A and B, where the curve meets the greatest and least distances at right angles, are called the *Apsides*.

If $r'' = r'$, then $u = \frac{1}{a}$, since $e = 0$; hence, the radius vector drawn from T is every where of the same value, or the curve is a circle. In this case, $h = \sqrt{[\mu a \,.\, (1 - e^2)]} = \sqrt{\mu a}$.

The quantity h, at a given distance from the centre, depends on the angle and the velocity of projection: let ϵ be the angle TLM, and V the velocity, then



$$V = \frac{Lp}{dt} = \frac{1}{dt} \times \frac{np}{\sin. \epsilon},$$

$$= \frac{dv}{u \,.\, dt \,.\, \sin. \epsilon} = \text{(see p. 14.)} \; \frac{hu}{\sin. \epsilon};$$

$$\therefore h = \frac{\sin. \epsilon \times V}{u}.$$

Hence, when a circle is described, $\epsilon = 90^\circ$, and $V = hu = \frac{h}{a} = \sqrt{\frac{\mu}{a}}$, is the velocity of a body describing a circle, of which the radius is a.

The equation

$$u = \frac{1}{a.(1-e^2)} \times [1 + e.\cos.(v-\pi)]$$

is the same as that which belongs to an *ellipse*, of which $2a$ is the major axis, ae the eccentricity, $v-\pi$ the angle LTB (see fig. p. 25.) and $\frac{1}{u} = TL$ the radius vector. Consequently the curve described by a body revolving round another, and attracted to it by a force varying inversely as the square of the distance is, generally, an ellipse. And this is the second of what are called *Kepler's Laws.* Kepler discovered that the orbits of the planets were ellipses, and Newton (see *Princ.* Sect. III. Prop. xi. Lib. 1. and Prop. xiii. Lib. 3.) proved that they must necessarily be so.

The *third of Kepler's Laws,* by which the relation, between the periodic times of bodies revolving in ellipses, and the major axes of those ellipses, is determined, may easily be deduced from the second equation; thus

$$dt = \frac{dv}{h.u^2} = \frac{1}{hu^2} \times -\frac{a(1-e^2).du}{e\sin.(v-\pi)} = -\frac{\sqrt{[a.(1-e^2)]}.du}{e.\sqrt{\mu}.\sin.(v-\pi)\,u^2}:$$

But, since $\cos.(v-\pi) = \frac{1}{e}[au(1-e^2)-1]$,

$$\sin.(v-\pi) = \frac{\sqrt{(1-e^2)}}{e}\sqrt{[a^2e^2u^2-(1-au)^2]};$$

$$\therefore dt = -\sqrt{\frac{a}{\mu}} \times \frac{du}{u^2\sqrt{[a^2e^2u^2-(1-au)^2]}},$$

$$= \frac{1}{e\sqrt{\mu a}} \times \frac{r\,dr}{\sqrt{\left[1-\frac{1}{e^2}\left(1-\frac{r}{a}\right)^2\right]}}.$$

Now, $r\,dr$ is the fluxion or differential of $\frac{1}{2}r^2$,

and $\frac{1}{2}r^2 = \frac{a^2}{2}.\left(\frac{r}{a}\right)^2 = \frac{a^2}{2}\left[\left(1-\frac{r}{a}\right)^2 - \left(1-\frac{2r}{a}\right)\right] =$

$\frac{a^2e^2}{2} \times \frac{1}{e^2}\left(1-\frac{r}{a}\right)^2 - \frac{a^2}{2}\left(1-\frac{2r}{a}\right)$. Hence,

$$dt = \frac{a^{\frac{3}{2}} e}{2\sqrt{\mu}} \times \frac{d\left[\frac{1}{e^2}\left(1 - \frac{r}{a}\right)^2\right]}{\sqrt{\left[1 - \frac{1}{e^2}\left(1 - \frac{r}{a}\right)^2\right]}} +$$

$$\frac{a}{e\sqrt{\mu a}} \times \frac{dr}{\sqrt{\left[1 - \frac{1}{e^2}\left(1 - \frac{r}{a}\right)^2\right]}},$$

and integrating,

$$t = -\frac{a^{\frac{3}{2}} e}{\sqrt{\mu}} \sqrt{\left[1 - \frac{1}{e^2}\left(1 - \frac{r}{a}\right)^2\right]} + \frac{a^{\frac{3}{2}}}{\sqrt{\mu}} \times W,$$

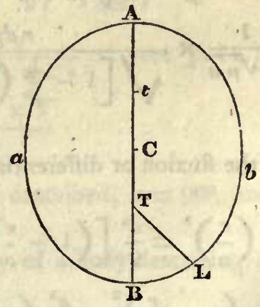
W being a circular arc of which the radius is 1, and the cosine is $\frac{1}{e}\left(1 - \frac{r}{a}\right)$.

Hence,

$$t = \frac{a^{\frac{3}{2}}}{\sqrt{\mu}} [W - e \sin. W].$$

No correction has been made to the integral of the above equation: t, therefore, and W are supposed simultaneously to equal nothing. But, if $W = 0$, its cosine $\frac{1}{e}\left(1 - \frac{r}{a}\right)$ must equal the radius 1; $\therefore r = a - ae = r' =$ (see figure,) TB. The time, then, in the preceding formula is reckoned from the apsidal and least distance TB.

The time of moving from B, the extremity of the least and



apsidal distance, to the *apside* A, or the time of half a revolution may be obtained by substituting in the preceding expression those values of W and sin. W, that correspond to the value of $r = r'' = a + ae$. Now, when $r = a + ae$, $\frac{1}{e}\left(1 - \frac{r}{a}\right)$, the cosine of W, $= \frac{1}{e} \times - e = -1$; $\therefore$ $W = 180^{0}$, and sin. $W = 0$, consequently, the time from B to A, or half the periodic time $= \frac{a^{\frac{3}{2}}}{\sqrt{\mu}} \times 180^{0}$, and the periodic time $(P) = \frac{a^{\frac{3}{2}}}{\sqrt{\mu}} \times 360^{0}$.

If $2a'$ were any other major axis, and P' the corresponding period, we should have

$$P' = \frac{a'^{\frac{3}{2}}}{\sqrt{\mu'}} \times 360^{0}\,*,$$

and, if $\mu = \mu'$, this analogy,

$$P : P' :: a^{\frac{3}{2}} : a'^{\frac{3}{2}},$$

which is the third famous Law discovered by Kepler to be true, and proved, on mechanical principles, to be so by Newton, in the 15th Proposition of the 3d Section of the *Principia*.

The preceding analogy is exactly true, when L as a material point revolves round T as a fixed centre: but this condition is merely hypothetical. In nature, L, representing a planet, is a body and revolves round T representing the Sun, another body. T,

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[*] If we would assume as known, by other methods, the area of the ellipse, we might arrive at this conclusion, (were that the sole object of investigation), by a shorter method. Thus, $dt = \frac{dv}{hu^2} = \frac{r^2\,dv}{h}$; $\therefore$ $t = \frac{1}{h}\int r^2\,dv$, but $\int r^2\,dv$ (estimating the whole of the area) $= 2$ area of ellipse $= 2\pi a^2\sqrt{(1 - e^2)}$, and h (see p. 25.) $= \sqrt{\mu a\,(1 - e^2)}$; $\therefore$ t, or the period, $= \frac{2\pi a^2\sqrt{(1 - e^2)}}{\sqrt{[\mu a\,(1 - e^2)]}} = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{\mu}} = 360^{0} \times \frac{a^{\frac{3}{2}}}{\sqrt{\mu}}$. The method, however, which is used in the text, leads to other results besides that of Kepler's Law, and is entirely independent of the ellipse and its properties.

therefore, since the attraction is mutual, cannot be fixed. If it be considered as the central body,* allowances must be made in the calculation for this supposition. When such allowance is made, μ is necessarily unequal to μ'. These quantities, as we shall hereafter see, represent the sums of the masses of the revolving and attracting bodies: for instance, if we wished to compare the periodic times of Jupiter and Mercury revolving round the Sun, and a and a' respectively represented the mean distances of those planets from the Sun, we should have

$$\mu = \text{Sun's mass} + \text{Jupiter's mass} = 1 + \frac{1}{1067.09},$$

$$\mu' = \text{Sun's mass} + \text{Mercury's mass} = 1 + \frac{1}{2025810},$$

and accordingly, instead of $\frac{P^2}{P'^2} = \frac{a^3}{a'^3}$, which Kepler's Law would give us, we have more exactly,

$$\frac{P^2}{P'^2} = \frac{\mu'}{\mu} \times \frac{a^3}{a'^3} = \frac{a^3}{a'^3} \times .99906 \text{ nearly.}$$

Newton, with a view of so correcting Kepler's Law that it should agree with the exact law that regulates the phenomena of Nature, composed the first Propositions of the 11th Section of his *Principia*.

The two equations, $r = \frac{a.(1-e^2)}{1+e.\cos.(v-\pi)}$, and $t = \frac{a^{\frac{3}{2}}}{\sqrt{\mu}}.[W - e.\sin.W]$, deduced from the differential equations of p. 21, afford us these two properties, 1st, that the curve described is an ellipse; and, 2dly, that the periodic times in different ellipses vary in the sesquiplicate ratio of the major axes. But, for Astronomical purposes, and with the view of comparing the results of calculation with observation, something beyond mere properties is required. The Astronomical Tables (see

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* If L should be equal to T, then, there would be no more reason to call L the revolving, and T the central body, than T the revolving, and L the central.

Astron. pp. 213, &c.) are constructed so as to assign the body's place at any given time; that can be done, if we know the value of r the radius vector, and its angular distance $v - \pi$, from the major axis: the position of the major axis, or, what amounts to the same, the place of the apside being supposed to be previously determined.

In order then to adapt the preceding equations to Astronomical uses, we must assign $v - \pi$ and r in terms of t; an operation which is purely analytical.

Let $a^{-\frac{3}{2}}\sqrt{\mu} = n$, then the two equations are

$$r = \frac{a.(1 - e^2)}{1 + e.\cos.(v - \pi)},$$

$$nt = W - e.\sin. W,$$

to these (see p. 28.) we may join a third equation,

$$r = a - ae \cos. W,$$

nt proportional to the time is called the *mean Anomaly*, $v - \pi$ the true anomaly, and W which is a subsidiary angle, and (see p. 28.) introduced for the purpose of expediting calculation, the *eccentric Anomaly*, (see *Astron.* Chap. XVIII.)

Since $nt = W - e \sin. W$, we shall have, by a known theorem *,

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* See *Trig.* pp. 213, &c.

Here x corresponds to nt, W to u, y to e, fu to $\sin. W$;

$\therefore X = fx = \sin. nt$, and $\frac{d(X^2)}{dx} = \frac{d(\sin.^2 nt)}{n\,dt} = 2\sin. nt.\cos. nt$

$= \sin. 2nt : \frac{d^2(X^3)}{dx^2} = \frac{d^2.(\sin^3 nt)}{n^2\,dt^2} = \frac{d^2}{4.n^2\,dt^2}[3\sin. nt - \sin. 3nt]$

$= -\frac{3}{4}\sin. nt + \frac{9}{4}\sin. 3nt$, &c. See also Lagrange, *Resolution des Equat. Numer.* ed. 1. pp. 234, &c. *Fonctions Analyt.* ed. 1. pp. 101. Laplace, *Mec. Celeste*, p. 117. Cousin, *Astron. Phy.* p. 15.

$$W = nt + e \,.\, \sin.\, nt + \frac{e^2}{1\,.\,2\,.\,2}\, 2 \sin.\, 2nt + \frac{e^3}{1\,.\,2\,.\,3\,.\,2^2}$$

$[3^2 \sin.\, 3nt - 3 \sin.\, nt] + \&c.$ and by the same theorem, since $\frac{r}{a} = 1 - e \,.\, \cos.\, W$,

$$\frac{r}{a} = 1 + \frac{e^2}{2} - e \,.\, \cos.\, nt - \frac{e^2}{2} \cos.\, 2nt - \&c.$$

and finally, making $\lambda = \frac{1}{1 + \sqrt{(1 - e^2)}}$,

$$v - \pi = W + 2\lambda \,.\, \sin.\, W + \frac{2\lambda^2}{2} \sin.\, 2W + \frac{2\lambda^3}{3} \sin.\, 3W + \&c.$$

$$= nt + \left(2e - \frac{e^3}{4} + \frac{5e^5}{96}\right) \sin.\, nt$$

$$+ \left(\frac{5}{4}e^2 - \frac{11e^4}{24} + \frac{17e^6}{192}\right) \sin.\, 2nt$$

$$+ \left(\frac{13e^3}{12} - \frac{43e^5}{64}\right) \sin.\, 3nt + \&c.$$

This is the direct analytical solution of the problem, in which it is proposed to find the true from the mean anomaly. It is, however, in practice, usually superseded by certain indirect and tentative methods of solution, which, by reason of the small eccentricities of the orbits of the planets, are found to be sufficiently exact, (see *Astron.* Chap. XVIII.) The problem, from its inventor, is called Kepler's (see *De motibus Stellæ Martis*, p. 300), and Sir Isaac Newton has given two solutions of it in the sixth Section of his *Principia.*

All the results that have hitherto been obtained depend essentially on this condition, that the force acting on the body L is *centripetal.* If that condition obtain, then Kepler's Law respecting the equable description of areas is true, whatever be the law of the variation of the force. But the two other laws, relating to the form of the orbit and the periodic time, require not only that the force should be centripetal, but that it should vary according to the inverse square of the distance; and, moreover, the third law is not exactly true, or the squares of the periodic times do not vary as the cubes of the mean distances, except we consider (which is contrary to the real circumstances in nature) the mass of the revolving to be evanescent relatively to that of the attracting or central body.

The case we have considered then is the simplest that can be imagined. Its conditions are the same as those in the third Section of the *Principia.* A material point L revolves, by virtue of a projectile motion and the agency of a centripetal force, round a fixed centre T; and describes in the same plane a curve called an Ellipse. The former circumstance, as we have seen, enabled us (see p. 21.) to reduce the differential equations to two, and to dispense with the consideration of a plane inclined to that of the body's orbit, and from which the latitude might be reckoned. By such simplification, the equation to the curve is arrived at independently of the theory of Projections. But, this object being obtained, it may, as we have already suggested (see p. 6.), be convenient to consider, besides the plane of the orbit, another plane, such as that of the ecliptic, and by means of which both the longitudes and latitudes of bodies are taken account of. This would oblige us to recur to the three differential equations of motion, (see p. 10.) and would introduce as arbitrary quantities the inclination of the two planes, and the angular distance of their intersection, usually called the Longitude of the Line of the Nodes.

These two new arbitrary quantities, the inclination of the planes, and the longitude of the line of the nodes, depend, it is plain, on the position of the orbit in space, and not on its nature and dimensions: for, they are created entirely by the introduction of a second plane. Of the other arbitrary quantities, a, e, and π, the two first, it is plain, depend on the nature of the curve, and the third on the position, in space, of the major axis.

The five quantities, the *major axis;* the *eccentricity;* the *longitude of the apside,* or, (when the Sun is the central attracting body), the *longitude of the perihelion;* the *inclination,* and the *longitude of the node,* are usually denominated the five *Elements* of a Planet's Orbit. To these may be joined a sixth, depending on the time at which the planet is in the perihelion of its orbit, and technically denominated the *Epoch of the Longitude* of the *Peri-*

helion. In the system of two bodies, one revolving round another fixed, and attracted towards it by a centripetal force, the elements of the elliptical orbit are invariable. This is easily inferred from what has preceded. But, for future purposes, it may be convenient to examine this matter more closely, and to exhibit the equations that involve the elements of the orbit. For such end, the following Chapter is principally intended.

CHAP. IV.

The Elliptical Elements of a Planet's Orbit determined: its Major Axis, Eccentricity, Longitude of the Perihelion, inclination of its Plane, Longitude of the Node, Epoch of the Passage of the Perihelion. The Elements of the Orbit considered as the Arbitrary Constant Quantities introduced by the Integration of the Differential Equations. Their invariability in the System of two Bodies. Expression for the Velocity in an Ellipse: in a Circle: in a Right Line, the Centripetal Force varying inversely as the Square of the Distance. Modification of the preceding Results, by considering the Masses of the Revolving and Central Body.

If we multiply the equation,

$$\frac{d^2 u}{d v^2} + u - \frac{\mu}{h^2} = 0,$$

by $d u$, then integrate and correct it by means of the condition of $\frac{d u}{d v} = 0$, when $u = \frac{1}{r'}$, there results

$$\frac{1}{2} \frac{d u^2}{d v^2} + \frac{1}{2} u^2 - \frac{\mu u}{h^2} - \frac{1}{2 r'^2} + \frac{\mu}{h^2 r'} = 0,$$

and consequently,

$$\frac{d u^2}{u^4} + \frac{d v^2}{u^2} = \left(2 \mu u + \frac{h^2}{r'^2} - \frac{2 \mu}{r'}\right) \times \frac{d v^2}{h^2 u^4}.$$

Now the left hand side of the equation is the square of the differential of the arc ($= L p^2$, see fig. p. 26.) and, since V^2 (the square of the velocity) $= \frac{L p^2}{d t^2} = L p^2 \times \frac{h^2 u^4}{d v^2}$, there results

$$V^2 = 2 \mu u + \frac{h^2}{r'^2} - \frac{2 \mu}{r'},$$

$$= 2 \mu u + \frac{2 \mu r''}{r' (r' + r'')} - \frac{2 \mu}{r'},$$

$$\left[\text{since, see p. 24. } \frac{\mu}{h^2} = \frac{1}{2}\left(\frac{1}{r'} + \frac{1}{r''}\right)\right]$$

$$= 2\mu\left(u - \frac{1}{r' + r''}\right),$$

$$= 2\mu\left(u - \frac{1}{2a}\right),$$

$$= \mu\left(\frac{2}{r} - \frac{1}{a}\right).$$

Hence, $a = \dfrac{\mu r}{2\mu - rV^2}$, and consequently, if V and r be given, a is determined, and must always remain invariable.

Hence also, which is a curious circumstance, a, the semi-axis major, is independent of the angle of projection: its value, at a given distance, depends solely on the velocity V.

If ϵ be the angle which the tangent makes with the radius vector r, or be the angle of projection, and if V be called the velocity of projection; then see p. 26.

$$V = \frac{h}{r \sin. \epsilon} = \frac{\sqrt{\mu a (1 - e^2)}}{r \sin. \epsilon},$$

and equating this value of V with the one just obtained, there results

$$\frac{\mu a . (1 - e^2)}{r^4 \sin.^2 \epsilon} = \mu . \left(\frac{2}{r} - \frac{1}{a}\right),$$

$$\text{or } \left(2r - \frac{r^2}{a}\right) \sin.^2 \epsilon = a(1 - e^2)^*,$$

whence ae, the eccentricity, $= a\sqrt{\left[1 - \left(\frac{2r}{a} - \frac{r^2}{a^2}\right)\sin.^2 \epsilon\right]}$,

Hence ae, like a, must remain invariable: but it is not, like a, independent of the angle of projection.

Since $r\left(= \dfrac{1}{u}\right) = \dfrac{a(1 - e^2)}{1 + e . \cos. (v - \pi)}$,

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* This agrees with Laplace's expression, *Mec. Celeste.* p, 191. obtained by a different process.

$$\cos.\,(v - \pi) = \frac{a\,(1 - e^2) - r}{r\,e};$$

consequently, the position of the *perihelion*, if a, e and r be given, is known.

In order to introduce the consideration of the other elements, the inclination and the longitude of the nodes, we must resume the three differential equations, which are

$$\frac{d^2 u}{d v^2} + u - \frac{\mu}{h^2\,(1 + s^2)^{\frac{3}{2}}} = 0 \ldots\ldots\ldots (a),$$

$$d\,t = \frac{d\,v}{h\,u^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (b),$$

$$\frac{d^2 s}{d v^2} + s = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (c),$$

in which u will not, as in the preceding Chapter, be $= \frac{1}{r}$, but $= \frac{1}{\rho}$.

Now this third equation is exactly similar to the one which we have already integrated (see p. 23.) we have, therefore,

$$s = A\sin.\,v + B\cos.\,v,$$

in which A and B must be determined, as a and b were, by the peculiar conditions of the case.

s is the tangent of the body's latitude, or, of its angular distance from the new plane, which is supposed to be inclined to the plane of the body's orbit. In some part, then, of the circuit of the orbit, the latitude, and consequently s, will $= 0$: at that point let $v = \theta$;

$$\therefore 0 = A\sin.\,\theta + B\cos.\,\theta,$$

$$\text{and } s = A\sin.\,v - A\,\frac{\sin.\,\theta}{\cos.\,\theta}\,.\cos.\,v,$$

$$= \frac{A}{\cos.\,\theta}\,(\sin.\,v - \theta).$$

In order to determine A, let $s = \gamma$ when $v - \theta = 90°$, then

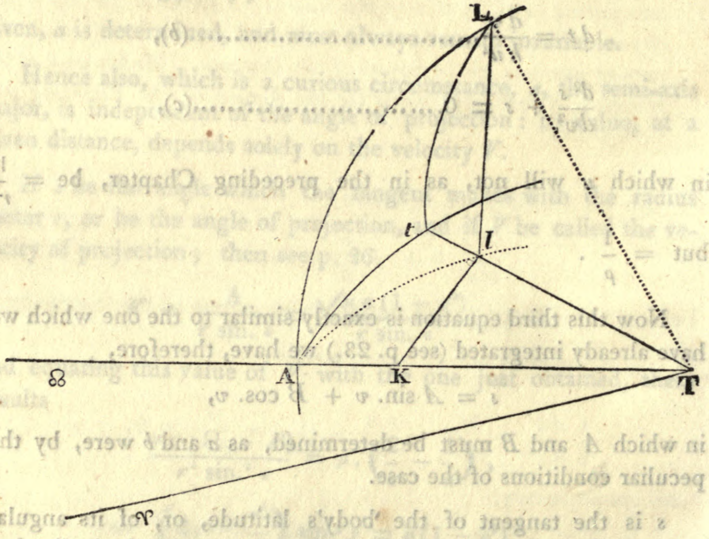
$$\gamma = \frac{A}{\cos.\ \theta} \times 1,$$

and consequently,

$$s = \gamma\ .\ \sin.\ (v - \theta),$$

which is the complete and exact integration of the third equation, and contains two arbitrary quantities γ and θ dependent on the inclination of the planes and the longitude of the node *.

Let *L* be the place of the body, *LA* part of the orbit, *A t* part of the circumference of the assumed plane, *TA* ☊ the intersection of the two planes, or the line of the nodes ; *T* ♈, an assumed line



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* The integral of the first equation $\frac{d^2 u}{d v^2} + u - \frac{\mu}{h^2 (1 + s^2)^{\frac{3}{2}}} = 0$,

$$\text{is } u = \frac{\mu}{h^2 (1 + \gamma^2)} \left[\surd (1 + s^2) + e\ .\ \cos.\ v\right]$$

$$= \frac{\mu}{h^2 (1 + \gamma^2)} \left[1 + \frac{\gamma^2}{4} + e\ .\ \cos.\ v - \frac{\gamma^2}{4} \cos.\ 2 v\right],$$

nearly, if γ be very small.

in the plane of At, from which the angular distance of the *projected* radius vector, or, its *longitude*, is measured: then the angle $AT\Upsilon = \theta$, $\Upsilon Tt = v$, and consequently $ATt = v - \theta$. Now Lt being perpendicular to At, we have, in the right-angled triangle LAt, by Naper's Rule, (See *Trig.* p. 136. ed. 2.)

$$1 \times \sin. At = \cot. LAt \times \tan. Lt,$$

$$\text{or, } \sin. (v - \theta) \times \tan. LAt = s,$$

the same equation as we have just obtained, since γ is that value of s which corresponds to $At = 90^0$, in which case Lt is the greatest latitude, and consequently measures the inclination of the two planes: γ, therefore, is the tangent of inclination, θ, as it is plain, is the longitude of the node.

It has already been shewn (see p. 15.) by the simple consideration of the action of the force, that the orbit described must lie in the same plane. And the same conclusion may easily be drawn from the equation $s = \gamma . \sin. (v - \theta)$; for, this is an equation between two sides and an angle of a right spherical triangle; the angle, which is invariable, denoting the inclination of the planes.

In the view we have taken of the subject, and by which our future operations will be regulated, the *elements* of a planet's orbit either algebraically depend on, or are themselves, the arbitrary quantities introduced by the integration of the differential equations. If the symmetrical equations (1), (2), (3), of p. 8. be integrated, then, since the integration of each equation will introduce two arbitrary quantities, the integration of all will introduce six. Of such quantities the elements of the orbit are *functions*. The integration of the equations (*a*), (*b*), (*c*), has produced, as we have seen, 5 arbitrary quantities, a, e, π, γ, θ; that is, the semi-axis major, the eccentricity *, the position of the perihelion, the tangent of the inclination, and the longitude of the node. The sixth arbitrary quantity or element is deficient, because the second equation ($u\,d^2 v - 2\,dv \,.\, du = 0$), has received only one integration, by which the differential equation $dt = \frac{dv}{hu^2}$, involving the arbitrary quantity h, was produced. If this equation then be integrated, there will result an equation such as

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* More correctly, its ratio to the semi-axis.

$$t + \epsilon = \int \frac{d\,v}{h\,u^2},$$

in which the arbitrary quantity, or correction ϵ, will determine the position of the body at a given epoch.

The integration of the original symmetrical equations (1), (2), (3), of p. 8, would exhibit, under a regular form, the arbitrary * quantities which it introduces; but then it would be necessary to combine those arbitrary quantities in order to obtain the values of the elements. The equations (a), (b), (c), produced by several operations from the former, are, as to their form, dissimilar: and consequently there would be no similarity of form between the arbitrary quantities introduced by their integration; but then, to balance this inconvenience, the quantities would express, almost exactly, the elements themselves.

The *invariability* of the elements has already been pointed out. It is peculiar to a system of two bodies; and, as such a system does not exist in nature, that is, as there is no planet which revolves round the Sun, and no *secondary* about its primary, undisturbed by the action of other heavenly bodies, the axes major, the eccentricities, the inclinations, the perihelia, and the places of the nodes of the planetary orbits may, for all that has hitherto appeared, be subject to change. The preceding equations, however, which involve the constant values of the elements, are not without their use, since they are preparatory to the investigation of the quantity and law of their variation.

Before we conclude this Chapter, we will notice some properties of a body in motion and acted on by a centripetal force varying inversely as the square of the distance.

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* *Arbitrary* quantities, such as a, b in p. 23, or A, B, in p. 37, are symbols assumed at pleasure, the particular values of which are to be fixed by the peculiar circumstances of the case. Thus $a \sin. v + b \cos. v$, where a and b are the arbitrary quantities, is the general form of solution both of the equation of p. 23, and of the equation (c), but the particular values of a and b, are, as we have seen (p. 23, 37.) quite different.

By the expression for the velocity in p. 36, we have

$$V = \frac{\surd[\mu a \, . (1 - e^2)]}{r \, . \sin. \epsilon}.$$

If $e = 0$, then (see p. 26.) $r = a$, and $\epsilon = 90^0$, the curve described being a circle; in this case then the velocity in a circle, of which the radius is a, is $= \sqrt{\frac{\mu}{a}}$. Similarly, the velocity in any other circle, of which the radius is r, is equal $\sqrt{\frac{\mu}{r}}$. Let this last equal U, then

$$V = \frac{U}{\sin. \epsilon} \, . \sqrt{\left[\frac{a}{r} \, (1 - e^2)\right]},$$

$$\text{or,} = \frac{U}{r \sin. \epsilon} \cdot \surd[a \, r \, (1 - e^2)],$$

which last expression agrees with what Newton asserts in Sect. III. Prop. xvi. Cor. 9. Since, $r \sin. \epsilon =$ perpendicular, and $a\,(1 - e^2) = \frac{a^2 - a^2 e^2}{a} = \frac{1}{2}$ *latus rectum* of an ellipse, of which a is the semi-axis major, and $a\,e$ the eccentricity.

$$\text{Since} \quad a = \frac{\mu \, r}{2\,\mu - r\,V^2}, \text{ and } U^2 = \frac{\mu}{r},$$

$$a = \frac{r\,U^2}{2\,U^2 - V^2}.$$

Hence, if $V^2 = 2\,U^2$, or if V, the velocity, should be to U, the velocity in a circle at the same distance, as $\surd 2 : 1$, a would be infinite. Now the ellipse, the major axis of which is infinite, has, at finite distances, the properties of a parabola; consequently, the velocity in a parabola $= U \surd 2 = \sqrt{\frac{2\,\mu}{r}}$.

If in the equation

$$V^2 = \mu \left[\frac{2}{r} - \frac{1}{a}\right],$$

we make $V = 0$, $2\,a = r$: and since (see p. 36.)

$$e = \sqrt{\left[1 - \left(\frac{2\,r}{a} - \frac{r^2}{a^2}\right) \sin.^2 \epsilon \right]},$$

e must $= 1$; consequently ae must $= a$, or the eccentricity must equal the semi-axis major. The ellipse, therefore, will be without breadth, and the centre of force, which is the focus of the ellipse, will coincide with one extremity of the axis major. At the other extremity, where the distance $r = 2a$, the velocity, as we have just seen, will equal nothing: at any other distance R, we shall have

$$V^2 = \mu\left(\frac{2}{R} - \frac{1}{a}\right),$$

$$= 2U^2 r \left(\frac{1}{R} - \frac{1}{2a}\right),$$

$$= 2U^2 \left(\frac{r}{R} - 1\right),$$

in which, U is the velocity in a circle the radius of which is r, and, in this case, equal to $2a$. If v be the velocity in the circle the radius of which is R, then, $\mu = U^2 . r = v^2 . R$, and consequently, $U^2 = v^2 . \frac{R}{r}$: whence,

$$V^2 = 2v^2\left(1 - \frac{R}{r}\right),$$

and, if this equation be expanded into a proportion, we have

$$V^2 : v^2 :: r - R : \frac{r}{2},$$

which, since r is the distance from which the body begins to fall towards the centre of force, is the same proportion as that which Newton has given in Sect. 7. Prop. 33.

If $V = v$, $r - R = \frac{r}{2}$; $\therefore R = \frac{r}{2}$, or the body in its rectilinear descent towards the centre of force acquires, during the first half of it, a velocity exactly adequate to the description of a circle, the radius of which is equal to half the body's original distance from the centre of force.

These results belong to a system of two bodies, or rather of one material point revolving round another assumed as a central point and the source of attraction. As the system of two bodies is, strictly speaking, hypothetical, and a mathematical simplification of the real system, or of the system of nature, the results are not exactly true; but they are nearly true, since the perturbations of the system are very small.

The alterations to be made in order to adapt the preceding results to a system, in which, instead of two points, one fixed and central, the other revolving, two masses of mutually attracting matter, and, therefore, neither quiescent, should be introduced, are very simple. If M represent the mass of T, or the number of attracting particles in that body, then, since each particle would attract a particle in L, or a corpuscle at L, with a force equal $\frac{1}{r^2}$ (r being equal LT), the aggregate attraction will be $\frac{M}{r^2}$. And if L should consist of several corpuscles, each, (since the attractive force of M is not supposed to be diminished by being exerted,) would be attracted by the same force $\frac{M}{r^2}$: consequently the *body* L would be attracted to T by the same force. But L itself attracts, and, if m should represent its mass, would attract each and every particle of the body T, and, consequently, the body T, with a force equal to $\frac{m}{r^2}$. Hence, L and T would be attracted towards each other by a force equal $\frac{M}{r^2} + \frac{m}{r^2}$, and the *approach* of these two bodies, inasmuch as it arises from the attraction, would be mutual, and even equal if L should equal T. But we may *suppose* one, the larger, T, for instance, to be at rest, and L to be attracted towards it by a single force $\frac{M+m}{r^2}$ equal the sum of the attractions: for, then, the mathematical consequence of the approach of L to T would be the same. This, in fact, is no other than the application of a common principle in Mechanics, namely, that, of the relative motion of the parts of a system continuing the same, when equal impulses and in the same directions are imparted to them. To reconcile, therefore, the case before us with that principle, we must suppose a force equal to $\frac{m}{r^2}$ simultaneously to be impressed on L and T, and in a direction measured from L towards T. The latter body, then, in consequence of the attraction of L equal to $\frac{m}{r^2}$ and *towards* L, and of this last communicated impulse equal to $\frac{m}{r^2}$ and *from* L, will, by the counteraction of equal forces, be at rest; and L, by their co-operation, will be urged towards T by the sum of attractions.

This principle and its consequences are not confined to the case in which L and T should be in motion solely from the agency of their mutual attraction. Should they possess any rotatory motion round any point, such as their centre of gravity, we may still *consider* T to be at rest, by *hypothetically* imparting to it and to L, a motion equal to its own, and contrary to its direction.

The results then of the preceding Chapters would hold true in a system of two bodies, if the symbol μ (see p. 6.) represented the sum of their masses. And we may now see why Kepler's Law is not exactly true, and what quantity of correction it requires (see p. 30.).

In order to draw the conditions of our mathematical theory nearer to the true conditions of nature, we have substituted attractive masses instead of points, and modified the results. The modification is a very slight one. But the next departure from the simple mathematical system of two bodies will lead us into investigations of very considerable intricacy. For, in nature, there is no heavenly body that is acted on solely by one body: it is always subjected to the action of a third; and, in fact, to the action of every body in the universe. Now, if we take the system of three bodies, and wish to determine the laws of the motion of one of these subjected to the action of the other two, we shall find the exact solution of the problem impossible; that is, beyond the powers, in their present state, of all the methods of calculation, whether they be analytical or geometrical. On this account, we must be contented with an approximate solution: and this we shall be able to obtain; since, in every case which nature presents to us of three or more bodies, the actions of the third and fourth body, are, by reason of their remoteness, very small in *disturbing* the motion of the second round the first considered as the centre.

This circumstance of the minuteness of the third body's action, explains why the investigation of the case of two bodies is made to precede that of three or more. The first investigation is preparatory to the latter; and its results, with slight modifications, belong to it. An exact ellipse is described when there are two bodies, and a curve differing little from an ellipse is described, when there are three or more bodies. The coincidence of the curve described in the first instance with a simple curve of known properties, has been the cause why the *elliptical* motion of a planet is considered as its *natural and proper* motion, and, accordingly, (that there might be no incongruity in language,) why a third is called a *disturbing* body: for its action obstructs the operation of the laws of elliptical motion. This, however, as it is plain, is mere mathematical fiction and contrivance: an ellipse is not the curve that is really described, but that curve is described, the equation of which is assigned by the solution of the *problem of the three bodies.* To arrive at that solution we make a stage at the problem of two bodies, not because it is necessary, but because it is mathematically convenient.

In the next Chapter we will consider the general effects of the *disturbing* force of a third *external* body: and, investigate an expression for its value. The way will thus be, in some degree, prepared for the solution of the *problem of the three bodies.* For, such is the title attached, for distinction's sake, to the investigation, in which it is required to find the laws of motion, the form of the orbit, &c. of a body in motion round one attracting body and disturbed, relatively to that motion, by the attraction of a third remote body: and it was under such form that Clairaut first stated the Lunar Theory, (see *Mem. Acad. Paris*, 1745. p. 329: and *Theorie de la Lune*, p. 3. ed. 2.) *.

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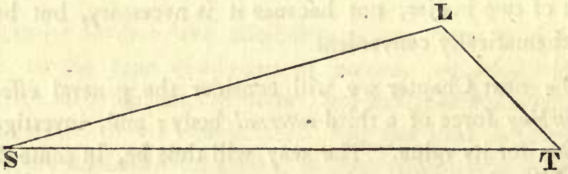
* Here properly ends the elliptic theory.

CHAP. V.

THEORY OF PERTURBATIONS.

A third attracting Body introduced into the System of two Bodies. Its Effects in disturbing the Laws of Motion and the Elements of that System. Expressions of the Values of the resolved Parts of the disturbing Force; the Ablatitious; the Addititious: the Force in the direction of the Radius Vector; the Tangential Force; Effects of these Forces in altering Kepler's Laws, &c. Approximate Values of the Forces when the disturbing Body is very remote. Expressions for the Forces, in the Problem of the three Bodies, by means of the Partial Differentials of a Function of the Body's Parallax, Longitude and Latitude.

In the preceding cases, we have supposed L to revolve round T, by virtue of some projectile motion oblique, in its direction, to LT, and of a centripetal force always urging L towards the point T.



The centripetal force was supposed to arise from an attraction resident in the component particles of the bodies L and T, and to be proportional to the number of such particles; or, in other words, to the masses of the bodies L and T. m and M, therefore, expounding those masses, and the law of the force being according to the inverse square of the distance (r), the centripetal force was represented by $\frac{M+m}{r^2}$.

If we illustrate these reasonings by reference to the Phenomena of Nature, and suppose L to represent the Moon, and T

the Earth, then, in consequence of their mutual attraction, the Moon, as has been shewn, will describe an ellipse round the Earth, and areas proportional to the time, if we suppose all foreign agency, or the attraction of the other bodies and planets of the system, to be abstracted.

In like manner, areas exactly proportional to the time, and an exact ellipse lying in the same plane, will be described, if L should represent any one of the planets, and T the Sun: or if L should represent a satellite, and T its primary: this essential condition being, in all cases, observed, namely, that the attraction of no other planet or satellite should interfere with the centripetal force.

But, it is plain, this condition can never be observed. Every particle of matter is supposed to be endowed with an attractive quality; consequently, if L and T should represent the Moon and Earth, S representing the Sun, or Venus, or any other planet, would attract both L and T, and be attracted by them: which attraction, from *the circumstances under which it acts*, is called a *Disturbing Force*.

Let m' represent the mass of S, then in this system of three bodies, L, T, and S,

the attraction of L to $T = \frac{M}{LT^2}$,

of T to $L = \frac{m}{LT^2}$,

of T to $S = \frac{m'}{ST^2}$,

of S to $T = \frac{M}{ST^2}$,

of L to $S = \frac{m'}{SL^2}$,

and of S to $L = \frac{m}{SL^2}$.

The four latter forces *disturb* the elliptical motion of L, not with their whole quantities, but with their difference. Suppose, for instance, that S should be so distant from L and T, that the difference of their distances relative to the distances themselves, might be neglected, and that finite parts of lines drawn, from L and T towards S might be esteemed parallel: then, L and T would be *equally* urged towards S and in parallel directions: and consequently, the relative motion of L round T would not be *disturbed*: for the force of the attraction of S under these circumstances, would be the same as that of the communication of two equal impulses to L and T (see p. 43.).

This circumstance of an evanescent, or very minute difference of the forces by which L and T are drawn towards S, takes place in nature. If L and T should represent the Moon and Earth, and S should represent either Jupiter or Saturn, then, by reason of the great distances of those planets, L and T would, equally and in parallel directions, be urged towards S, and no perturbation, or, at most, a very slight one, of the laws of elliptical motion, and of the equable description of areas, would ensue.

Such a case as we have now instanced, does not *mathematically* belong to the problem of the three bodies, that is, does not require any special methods, or methods of solution beyond those that have been already used. It may, however, for the *sake of extending* a *classification*, be made to belong to it, and be considered as its most simple and limiting case.

Since a third body, so distant as Jupiter is relatively to the Moon and Earth, would produce no alteration in Kepler's Laws, we must enter on the real problem of the three bodies, by assuming a case in which the third body S although very, should not be so excessively, distant, but that lines drawn from it to L and T should, in some positions, be unequal in length, and inclined the one to the other.

Such a case, amongst many others, is that in which the Sun should be the central, Saturn the revolving, and Jupiter the disturbing body *. For when Saturn is in opposition, or Jupiter is between that planet and the Sun, his disturbing force is to the force of the Sun, (or the centripetal force by which Saturn is urged towards the Sun) as 1 to 211; since $\frac{\text{♃'s mass}}{(\text{♃'s dist. from ♄})^2}$ to $\frac{\text{☉'s mass}}{(\text{♄ dist.})^2}$ are nearly as those numbers.

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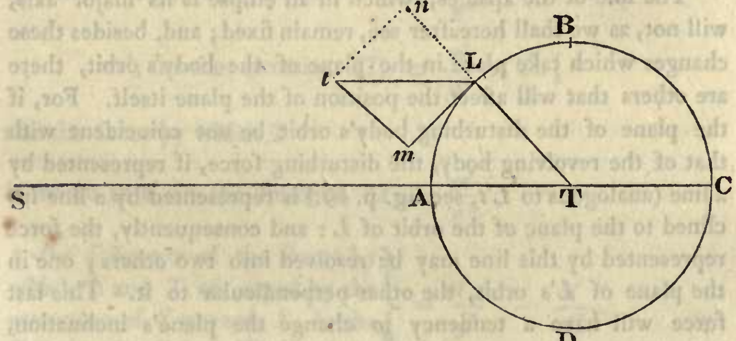
* Actio quidem Jovis in Saturnum non est omnino contemnenda. Nam, &c. et hinc oritur perturbatio orbis Saturni in singulis planetæ hujus cum Jove conjunctionibus adeo sensibilis ut ad eandem Astronomi hæreant. Newton, p. 409. ed. 3. 1726.

In the preceding position of Jupiter, the effect of his disturbing force would be merely to increase the *centripetal* force of the Sun, and consequently, the equable description of areas would not be disturbed. But this would not happen in other positions of Jupiter. In fact, whenever the *third* body is so near as either to diminish or augment, in any position, the central force by which the revolving body is urged, it must, in other positions, produce a disturbing force oblique in its direction to the radius, and consequently (see p. 19.) tending to interrupt the equable description of areas. For, if S be the third body, and m' its mass, its disturbing force on L, when L is at A, is $\frac{m'}{SA^2} - \frac{m'}{ST^2}$; and, consequently, can have none effect, except (see p. 48.) the difference of SA and ST, bear, (when it is, according to the conditions of the case, numerically expounded) some sensible proportion to SA and ST. But if that be the case, then, at points, such as L, intermediate between A and B, the difference of ST and SL will be of some sensible magnitude, and consequently the difference of the forces by which L and T are urged to S, and on which the disturbing force depends, will be of some magnitude. That dif-



ference may be represented by a line $L\,t$ either parallel to ST, or nearly so, and consequently will always (except at four points) admit of being resolved into two other lines representing forces, one in the direction of the radius LT, the other in a direction perpendicular to LT. It is this last force (see p. 19.) which prevents the operation of Kepler's Law of the equable Description of Areas.

This is not the sole effect of a disturbing force; for since, in some positions (see p. 49.), it augments, and in others diminishes* the centripetal force, and not according to the law of its variation, the law of the resulting force, by which L is urged, is not that which it was supposed to be, when (see p. 27.) the orbit described by L was proved to be an ellipse.

The line of the apsides, which in an ellipse is its major axis, will not, as we shall hereafter see, remain fixed; and, besides these changes which take place in the plane of the body's orbit, there are others that will affect the position of the plane itself. For, if the plane of the disturbing body's orbit be not coincident with that of the revolving body, the disturbing force, if represented by a line (analogous to $L\,t$, see fig. p. 49.) is represented by a line inclined to the plane of the orbit of L: and consequently, the force represented by this line may be resolved into two others; one in the plane of L's orbit, the other perpendicular to it. This last force will have a tendency to change the plane's inclination, and also (when combined with the body's motion) to change the position of its intersection with another plane (such as the ecliptic), or, which is the same thing, to change (see *Astronomy*, p. 40.) the longitude of the nodes.

Such are, on general grounds, the discernible effects of a disturbing force; they will, perhaps, be more distinctly perceived from its mathematical expression, which we shall now proceed to investigate: and first, for the sake of simplicity, we will suppose the planes of the orbits of the revolving and disturbing bodies to be coincident.

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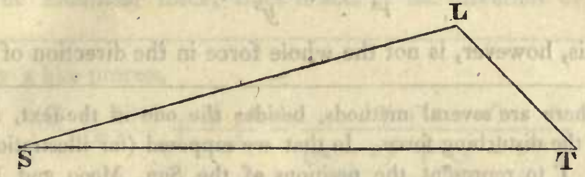
* If Saturn be in conjunction, then Jupiter attracting the Sun more than it does Saturn has the effect of *diminishing* the centripetal force.

Let, $LT = r$, $ST = r'$, $SL = y$, and the angle $STL = \omega$, let also M, m, m', be, respectively, the masses of the bodies placed



at T, L, and S; T (the Earth) being considered to be the central body round which L (the Moon), disturbed by the action of S (the Sun), is supposed to revolve; then, the force by which

L is drawn to S, in the direction LS, $= \dfrac{m'}{y^2}$,

and, in the direction ST, $= \dfrac{m'}{y^2} \times \dfrac{r'}{y}$,

and the force by which T

is drawn to S, in the direction ST, $= \dfrac{m'}{r'^2}$;

$$\left.\begin{array}{l}\therefore \text{ the difference of the forces by} \\ \text{which } L \text{ and } T \text{ are urged by the} \\ \text{attraction of } S \text{ towards } S\text{, in the} \\ \text{direction parallel to } ST\end{array}\right\} = \frac{m' r'}{y^3} - \frac{m'}{r'^2}.$$

$$\left.\begin{array}{l}\text{The remaining resolved part of} \\ \text{the force of } S \text{ on } L \text{ lies in the} \\ \text{direction } LT\text{, and}\end{array}\right\} = \frac{m'}{y^2} \times \frac{r}{y}.$$

This latter force always acts in the direction LT, and, since it increases the centripetal, which is reckoned the chief force, it is technically called the *Addititious* Force. The centripetal force in this case, since T is supposed to be fixed, must be represented by the sum of the attractions of L to T and of T to L (see p. 43.), and therefore, by

$$\frac{M + m}{r^2},$$

consequently, the compounded force of *L* to *T* (which is in fact, a centripetal force) is, now,

$$\frac{M+m}{r^2} + \frac{m'r}{y^3}\text{*}.$$

This, however, is not the whole force in the direction of *LT*;

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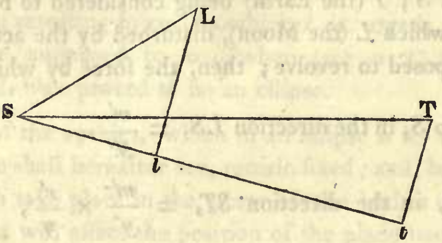
* There are several methods, besides the one in the text, for resolving the disturbing force. In that we supposed (for illustration), *S*, *L*, and *T* to represent the positions of the Sun, Moon and Earth. Suppose now *S* still to represent the Sun, but *L* and *T* Jupiter and Saturn, then *S* is the central, *L* may be the revolving, and *T* the disturbing body; or, *T* may be the revolving, and *L* the disturbing: take the former case, and let the symbols for these bodies be used to denote their masses: let *Sl*, *St* be denoted by *x*, *x′*, *Ll*, *Tt*, by *y*, *y′*, *SL*, *ST*, by *r*, *r′*, *Tt*, *Ll* being perpendicular to *St*: then the force by which *L* is drawn to *S*, in the direction of *Sl*, by the mutual attraction of *L* and *S*, is (see p. 51.)



$$\frac{\odot + \text{♃}}{r^2} \cdot \frac{x}{r}.$$

and the forces by which *T* draws *S* and *L* in the same direction are, respectively,

$$\frac{\text{♄}}{r'^2} \cdot \frac{x'}{r'} \text{ and } \frac{\text{♄}}{(x'-x)^2+(y'-y)^2} \times \frac{x'-x}{\sqrt{[(x'-x)^2+(y'-y)^2]}};$$

but *L* is disturbed in this direction by the *difference* only of these latter forces, consequently by

$$\frac{\text{♄}\,.\,x'}{r'^3} - \frac{\text{♄}\,.\,(x'-x)}{[(x'-x)^2+(y'-y)^2]^{\frac{3}{2}}},$$

and accordingly the whole force acting on *L* in the direction parallel to *x*, is

$$\frac{(\odot + \text{♃})x}{r^3} + \frac{\text{♄}\,.\,x'}{r'^3} - \frac{\text{♄}\,.\,(x'-x)}{[(x'-x)^2+(y'-y)^2]^{\frac{3}{2}}},$$

By

there still remains to be added to it, a resolved part of that other force, which acts in a direction parallel to *ST*.

The *addititious* force, since it acts in the direction of a line

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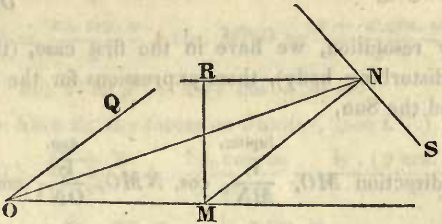
By a like process,

$$\frac{(\odot + \text{♄})x'}{r'^3} + \frac{\text{♃} \cdot x}{r^3} - \frac{\text{♃} \cdot (x - x')}{[(x - x')^2 + (y - y')^2]^{\frac{3}{2}}},$$

expresses the force in a direction parallel to x by which Saturn is acted on, when it revolves round the Sun, and is disturbed by Jupiter: and these two expressions are the same as what M. Laplace uses in his Theory of Jupiter and Saturn. (See *Memoirs of the Academy of Sciences*, 1785. pp. 38, &c.)

The foregoing, as we have said, is, very nearly, Laplace's method of valuing the forces. For the purpose of farther *illustration*, and solely for such purpose, Euler's, that which he gave in the 7th Vol. of the *Prix de l'Academie des Sciences*, 1769. is subjoined.

Let O be the place of the Sun ($\odot$), M of Jupiter (♃), N of Saturn (♄), let MR be perpendicular to OM, NS to MN, and let the symbols



$\odot$, ♃, ♄, denote the masses of the bodies they are meant to signify, then the forces attracting the Sun, Jupiter, Saturn,

are respectively,

in the directions	Sun.	Jupiter.	Saturn.
OM	$\frac{\text{♃}}{OM^2}$,	$\frac{\odot}{OM^2}$,	
ON	$\frac{\text{♄}}{ON^2}$,		$\frac{\odot}{ON^2}$,
MN		$\frac{\text{♄}}{MN^2}$,	$\frac{\text{♄}}{MN^2}$.

Now, as before, if the Sun be supposed fixed, and Jupiter be the body joining *L* the revolving, and *T* the central body, does not (see p. 15.) disturb the equable description of areas; but, since the expression for its value, $\frac{m'r}{y^3}$, is not of the form $\frac{A}{r^2}$, (*A* being

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body disturbed, we must make the Sun's force the sum of the forces by which the Sun and Jupiter mutually attract each other;

∴ the force urging Jupiter in the direction *OM* is $\frac{\odot + \text{♃}}{OM^2}$,

in the direction *MN* is $\frac{\text{♄}}{MN^2}$,

and the force urging the Sun in the direction *ON* is $\frac{\text{♄}}{ON^2}$.

If Saturn be the disturbed, and Jupiter the disturbing body, then

the force urging Saturn in the direction *ON* is $\frac{\odot + \text{♄}}{ON^2}$,

in the direction *MN* is $\frac{\text{♄}}{MN^2}$,

and the force urging the Sun in the direction *OM* is $\frac{\text{♃}}{OM^2}$.

Hence, by resolution, we have in the first case, (that is, when Saturn is the disturbing body), these expressions for the forces attracting Jupiter and the Sun,

	Jupiter.	Sun.
in the direction *MO*,	$\frac{\text{♄}}{MN^2}$ cos. NMO,	$\frac{\text{♄}}{ON^2}$ cos. MON,
in the direction *MR*,	$\frac{\text{♄}}{MN^2}$ sin. NMO,	$\frac{\text{♄}}{ON^2}$ sin. MON,

and when Jupiter is the disturbing body, we have these expressions for the forces attracting Saturn and the Sun,

	Saturn.	Sun.
in the direction *NO*,	$\frac{\text{♃}}{MN^2}$ cos. MNO,	$\frac{\text{♃}}{OM^2}$ cos. MON,
in the direction *NS*,	$\frac{\text{♃}}{MN^2}$ sin. MNO,	$\frac{\text{♃}}{OM^2}$ sin. MON.

hence collecting the forces, we have, when Saturn is the disturbing body, these expressions for the forces on Jupiter,

in

a constant quantity), in other terms, since it does not vary according to the law of the inverse square of the distance (r), it disturbs the elliptical form of the orbit *. The other force, $\left(\frac{m'r'}{y^3} - \frac{m'}{r'^2}\right)$ neither acts in the direction of the radius, nor varies according to the law of the inverse square of the distance: it disturbs therefore both the equable description of areas and the elliptical form

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in the direction MO, $\frac{\odot + \text{♃}}{OM^2} + \frac{\text{♄}}{ON^2} \cos. MON - \frac{\text{♄}}{MN^2} \cos. NMO$,

in the direction MR, $-\frac{\text{♄}}{ON^2} \sin. MON + \frac{\text{♄}}{MN^2} \sin. NMO$,

and for the forces soliciting Saturn,

in the direction ON, $\frac{\odot + \text{♄}}{ON^2} + \frac{\text{♃}}{OM^2} \cos. MON + \frac{\text{♃}}{OM^2} \cos. MNO$,

in the direction NS, $\frac{\text{♃}}{OM^2} . \sin. MON - \frac{\text{♃}}{MN^2} \cos. MNO$.

Let $OM = x$, $ON = y$, $MN = z$, $\angle MON = \omega$,

then,

$$\sin. NMO = \frac{y . \sin. \omega}{z}, \quad \cos. NMO = \frac{y \cos. \omega - x}{z},$$

$$\sin. MNO = \frac{x . \sin. \omega}{z}, \quad \cos. MNO = \frac{y - x \cos. \omega}{z},$$

$$\text{for, } z^2 = y^2 - 2xy \cos. \omega + x^2.$$

Hence, we have for the forces on Jupiter, (see l. 7.)

(direction MO,) $\frac{\odot + \text{♃}}{x^2} + \frac{\text{♄} . \cos. \omega}{y^2} - \frac{\text{♄} . (y \cos. \omega - x)}{z^3}$,

(direction MR.) $-\frac{\text{♄} . \sin. \omega}{y^2} + \frac{\text{♄}\, y \sin. \omega}{z^3}$,

and on Saturn,

(direction NO,) $\frac{\odot + \text{♄}}{y^2} + \frac{\text{♃} \cos. \omega}{x^2} + \frac{\text{♃} (y - x \cos. \omega)}{z^3}$,

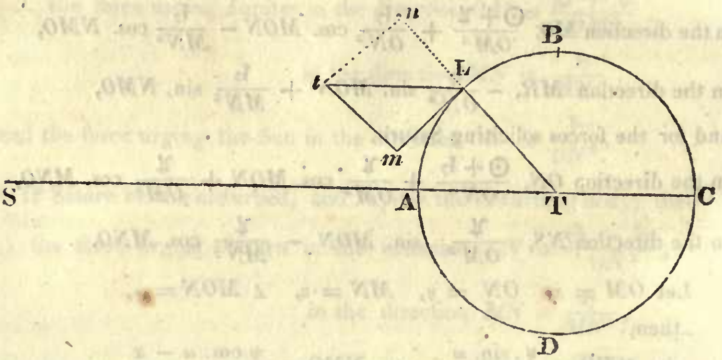
(direction NS,) $\frac{\text{♃} \sin. \omega}{x^2} - \frac{\text{♃}\, x \sin. \omega}{z^3}$.

See also, on this point, Robison's *Mechanical Philosophy*, pp. 377. 395.

* This, perhaps, is too *inconsequential*: for, it may be contended, there are no simple and palpable reasons (short of those derived from demonstration) for supposing that an ellipse cannot be described with two laws of force.

of the orbit; it renders, in fact, irregular the operation of two of Kepler's Laws. The equable description of areas, however, is not disturbed by the whole of this force, but by that part of it (see p. 19.) which acts perpendicularly to the radius. The value of that part we shall now proceed to find.

Produce TL, and on it, from t, let fall the perpendicular tn, then if Lt represent the whole of the force, $\left(\frac{m'r'}{y^3} - \frac{m'}{r'^2}\right)$, tn



will represent that part of it which acts perpendicularly to the radius, and Ln the part which acts in the direction of the radius.

If $ABCD$ be a circle, and Lm $(=tn)$ be a tangent at the point L, Lm will be perpendicular to the radius LT, and consequently parallel to tn. The force, therefore, which acts perpendicularly to the radius acts in the direction of the tangent, and consequently it may be denominated the *Tangential* force.

The tangential force

$$(tn = Lt \times \sin. nLt) = m'\left(\frac{r'}{y^3} - \frac{1}{r'^2}\right) \sin. \omega.$$

The resolved part of the force in the direction of radius

$$(Ln = Lt \times \cos. nLt) = m'\left(\frac{r'}{y^3} - \frac{1}{r'^2}\right) \cos. \omega.$$

The former of these is the force T of p. 19. and since it never is nothing, except when $\sin. \omega$ is $= 0$, or $\frac{r'}{y^3} = \frac{1}{r'^2}$, the equable description of areas is every where disturbed in the orbit $ABCD$, except at the points A, C, or at points near B, D; that is, except (see *Astronomy*, p. 43.) the body L be in *syzygies*, or near *quadratures*.

We may now express the whole of the forces that act on the body L.

The whole force in the direction of the radius, or (see p. 10.)

$$P = \frac{M+m}{r^2} + \frac{m'r}{y^3} + m'\left(\frac{1}{r'^2} - \frac{r'}{y^3}\right)\cos.\omega.$$

The tangential force, or $T = m'\left(\frac{r'}{y^3} - \frac{1}{r'^2}\right)\sin.\omega$ *.

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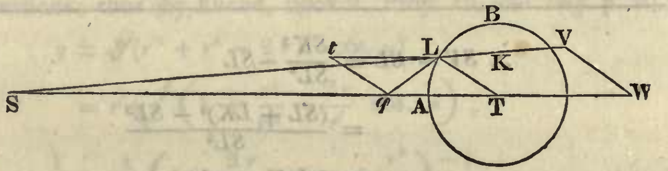
* Newton, to whom we owe the complete Theory of two Bodies, and the first Essays towards the Theory of three Bodies, has, on principles the same as the preceding, but by a different process, investigated the expressions for the disturbing forces. In the 11th Section he has expressed, by means of constructions and lines, what, in the text, has been expressed by symbols. According to him, SK is assumed to represent the attraction of L towards S, at the mean distance SB; then, on that supposition, ST will represent the attraction of T towards S; and $SV = SK \times \frac{SK^2}{SL^2}$ (the force varying inversely as the square of the distance) will represent the attraction of L towards S, at the distance SL. This force SV may be resolved into VW parallel to LT, and SW. The former, acting parallel to LT and from L towards T, increases the attraction of L to T: the latter draws L towards S: but, T is also, by the force ST, drawn towards S; and, the difference by which L is more drawn towards S than T, is TW: on this the *perturbation* of the system chiefly depends: but it is the force KV, when it acts in syzygies, that Newton calls the *Ablatitious*: VW he calls the *Addititious* (see p. 51.).



Newton's Commentators thus find an approximate expression for TW, on the supposition that S is very distant,

$$SV = \frac{SK^3}{SL^2};$$

$\therefore SV$

The former is compounded partly of the centripetal and partly of the disturbing force, but the latter entirely originates from the disturbing force.

The preceding are the expressions for the forces when the disturbing body moves in an orbit exterior to that of the revolving body; as in the case of the Sun disturbing the Moon's elliptical motion round the Earth. But there are cases when the orbit of the *third* body is within that of the disturbed; as in the instance of Venus disturbing the Earth's motion round the Sun, and in that of Jupiter disturbing Saturn's. The sole alteration which this change of condition will introduce into the preceding expressions, is a change of sign in the perpendicular or tangential force.

Having now obtained expressions for P and T, we might substitute them in the differential equations of p. 10, and then the sole difficulty of finding the form of the orbit, the variation of

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$$\therefore SV - SL = \frac{SK^3}{SL^2} - SL$$

$$= \frac{(SL + LK)^3 - SL^3}{SL^2}$$

$$= 3\,L\,K, \text{ nearly.}$$

Lt, therefore, drawn parallel to TW, and $=3LK$, represents, nearly, the force TW; it equals $3\,LT \times \cos. ATL$ and it may, (as it has been already done), be resolved into two other forces, one in the direction of the radius LT, the other perpendicular to such direction.

From this last construction we may proceed to one which Robison (see *Elements of Mechanical Philosophy*, p. 377.) has made. Draw tq parallel to LT, then since it represents, nearly, the addititious force, Lq, compounded of Lt, tq, represents the *whole* of *the disturbing* force.

From this construction and the value of Lq, Robison finds, when the orbit of L is inclined to that of S, that part of the disturbing force which acts perpendicularly to the plane (see p. 50.).

With the text and the notes (see pp. 51, 52, &c.) we have now abundant illustration of the method of resolving the disturbing force. In principle, or philosophically viewed, the instances are not materially different. But they will serve as exercises to the Student, and, in that way, tend to familiarize his mind with an abstruse subject.

the time, &c. would consist in the analytical solution of the resulting equations. But, previously to encountering this difficulty, (and it is no inconsiderable one), it may be expedient to consider some general effects which are deducible from the preceding expressions for the disturbing forces.

Whatever be the application of the *problem of the three bodies*, there are always circumstances in the case that lessen the practical difficulties of the analytical solution. When Venus disturbs Jupiter's motion, or the Sun disturbs the Moon's, the radius of the orbit of the disturbing body is small, in one case, and, in the other, large with respect to the radius of the orbit of the disturbed body; let us take the latter case, and make r' large (it nearly $= 400\,r$), then, instead of $\frac{1}{y^3}$, we are enabled to substitute its *approximate* value, and, thereby, to render much more simple the preceding expressions; thus (by Euclid, Book 2. Prop. 12, and *Trig*. p. 20.)

$$y = \sqrt{(r'^2 + r^2 - 2\,r\,r' . \cos . \omega)}$$

$$= r'\sqrt{\left(1 + \frac{r^2}{r'^2} - \frac{2\,r}{r'} \cos . \omega\right)};$$

$$\therefore \; \frac{1}{y^3} = \frac{1}{r'^3}\left(1 - \frac{2\,r}{r'} \cos . \omega + \frac{r^2}{r'^2}\right)^{-\frac{3}{2}}$$

$$= \frac{1}{r'^3}\left(1 + \frac{3\,r}{r'} \cos . \omega\right), \text{ nearly,}$$

that is, this is the value of $\frac{1}{y^3}$, when in the expanded expression, the terms involving $\frac{r^2}{r'^2}$, and higher powers of $\frac{r}{r'}$ are rejected.

Hence, substituting in the expressions of p. 57.

$$P = \frac{M+m}{r^2} + \frac{m'\,r}{r'^3}\left(1 + \frac{3\,r}{r'} \cos . \omega\right)$$

$$- \frac{3\,m'\,r}{r'^3} \cos . \omega \times \cos . \omega$$

$$= \frac{M+m}{r^2} + \frac{m'\,r}{r'^3} - \frac{3\,m'\,r}{r'^3}\left(\frac{1}{2} + \frac{\cos . 2\,\omega}{2}\right),$$

$$= \left(\text{by } Trig. \text{ p. 36. and by rejecting the term involving } \frac{3\,r^2}{r'^4}\right)$$

$$\frac{M+m}{r^2} - \frac{m'\,r}{2\,r'^3} - \frac{3\,m'\,r}{2\,r'^3}\cos.\,2\,\omega\,^*.$$

Similarly, $T = \frac{3\,m'\,r}{r'^3}\cos.\,\omega\,.\,\sin.\,\omega$,

$$= \frac{3\,m'\,r}{2\,r'^3}\,.\,\sin.\,2\,\omega \quad (Trig.\ \text{ed. 2. p. 36.})$$

Let $M + m = \mu$, then at the points A and C, when the body is in syzygy,

$$P = \frac{\mu}{r^2} - \frac{2\,m'\,r}{r'^3},$$

and the disturbing force $\left(P - \frac{\mu}{r^2}\right) = -\frac{2\,m'\,r}{r'^3}$.

At the same points, T, the tangential disturbing force, $= 0$.

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[*] When r is the larger quantity, or when the disturbing body is *within* the orbit of the revolving body (as in the case of the Moon disturbing the Sun's or Earth's motion), the expression for P will differ from the preceding, although the expanded form for $\frac{1}{y^3}$ will be similar in both cases: for when r is the larger quantity,

$$\frac{1}{y^3} = \frac{1}{r^3}\left(1 + \frac{3\,r'}{r}\cos.\,\omega\right);$$

$$\text{but } P = \frac{M+m}{r^2} + \frac{m'\,r}{r^3} + \frac{3\,m'\,r'}{r^3}\cos.\,\omega,$$

$$+ \frac{m'\cos.\,\omega}{r'^2} - \frac{m'\,r'}{r^3}\left(1 + \frac{3\,r'}{r}\cos.\,\omega\right)\cos.\,\omega,$$

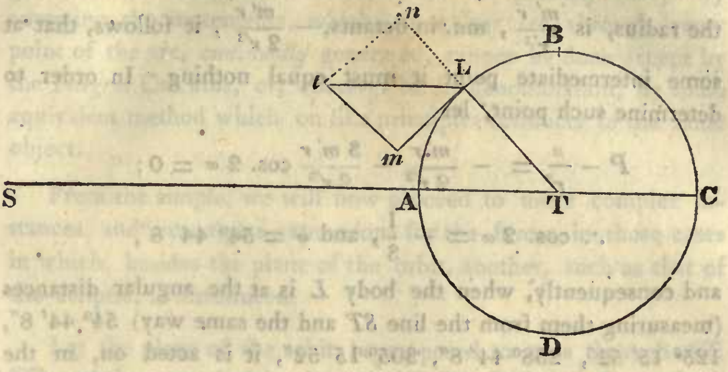
$$= \frac{M+m+m'}{r^2} + \frac{m'\,.\,\cos.\,\omega}{r'^2} + \frac{2\,m'\,r'}{r^3}\,.\cos.\,\omega + \&c.$$

neglecting the terms involving $\frac{1}{r^4}$, &c., and if besides these we neglect, on the supposition that the disturbing mass m' is very small, $\frac{m'}{r^2}$, $\frac{m'}{r^3}\cos.\,\omega$, the expression for the whole force in the direction of the radius is reduced to this

$$P = \frac{M+m}{r^2} + \frac{m'\,.\cos.\,\omega}{r'^2},$$

and by a similar computation T is reduced to $+\frac{m'}{r'^2}\sin.\,\omega$.

At the points B and D, that is, when the body is in quadrature,



$$P = \frac{\mu}{r^2} + \frac{2\,m'\,r}{2\,r'^3},$$

and the disturbing force $\left(P - \frac{\mu}{r^2}\right) = \frac{2\,m'\,r}{2\,r'^3}$. The centripetal force therefore is diminished in syzygy by the effect of the disturbing force and by the quantity $\frac{2\,m'\,r}{r'^3}$; and augmented in quadrature by $\frac{2\,m'\,r}{2\,r'^3}$, that is, by half the former quantity.

Since (see p. 29.) $\frac{m'}{r'^3} = \frac{M}{r^3} \times \left(\frac{☽\text{'s period}}{⊕\text{'s period}}\right)^2$,

$$\frac{m'\,r}{r'^3} = \frac{M}{r^2} \times .005595, \text{ or } = \frac{M}{r^2} \times \frac{1}{178.7},$$

or, since $\frac{M}{r^2}$ represents the Moon's gravity, the mean value of the addititious force is, nearly, $\frac{1}{179}$th of the Moon's gravity.

At the points B and D the tangential force is $= 0$.

The points exactly intermediate to the points of quadratures and syzygies are called *Octants*. At such points ω either $= 45^0$, or 135^0, or 225^0, or 315^0, and consequently (see *Trig.* p. 28.) $\cos. 2\,\omega = 0$, and $\sin. 2\,\omega = \pm\,1$. Therefore, at the octants,

$$P - \frac{\mu}{r^2} = -\,\frac{m'\,r}{2\,r'^3},$$

$$\text{and, } T = \pm\,\frac{3\,m'\,r}{2\,r'^3}.$$

This last is the greatest value of the tangential force.

Since in quadratures the disturbing force, in the direction of the radius, is $\frac{m' r}{r'^3}$, and, in octants, $-\frac{m' r}{2 r'^3}$, it follows, that at some intermediate point it must equal nothing. In order to determine such point, let

$$P - \frac{\mu}{r^2} = -\frac{m' r}{2 r'^3} - \frac{3 m' r}{2 r'^3} \cos. 2\omega = 0;$$

$$\therefore \cos. 2\omega = -\frac{1}{3}, \text{ and } \omega = 54^0\ 44'\ 8'',$$

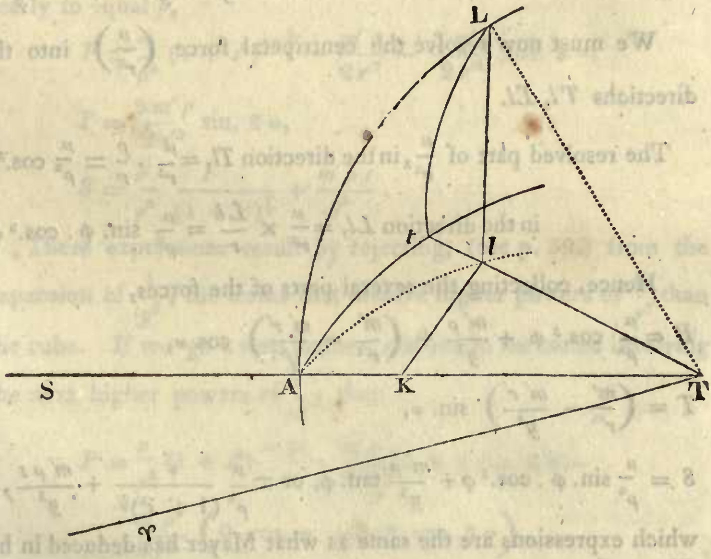
and consequently, when the body *L* is at the angular distances (measuring them from the line *ST* and the same way) 54° 44′ 8″, 125° 15′ 52″, 238° 44′ 8″, 305° 15′ 52″, it is acted on, in the direction of the radius, solely by the centripetal force of the body *T*.

The preceding are general inferences derived from the expressions for the disturbing force, on certain grounds and conditions; one of which is, the circular form of the orbit. And we might easily make farther inferences of the same kind, relative to the alteration in the curvature of the orbit and in the velocity, at the several points of syzygies, quadratures and octants. Such inferences, however, would be wholly of a general and indefinite nature, and besides, would necessarily be included in the results derived from mathematical investigation, and which, in the course of the Work, will be entered on.

In this investigation we must rely on the Integral Calculus, or on some equivalent method. This will be plain by attending to one instance, that of the alteration in the velocity produced by the disturbing force. At the octants (see p. 61.), the tangential force is in its maximum state. Its immediate effect, therefore, which is the increment of velocity, will there be the greatest. At the next point contiguous to the octants, the tangential force, and its exponent, the increment of the velocity, will be less: and, at all points between the octants and syzygies, the quantity of the tangential force and the increment of velocity thereby generated, will be successively less. At syzygies the tangential force will be nothing, but the velocity itself being the sum of the uniform velocity in the circular orbit, and of the *accumulation* of the increments of velocity generated by the accelerating force will be the greatest. We cannot, however, know its quantity except by summing the increments, which, since they are, through every point of the arc, *continually* generated, cannot be done except by the *Integral* Calculus, or, whatever be its denomination, by some equivalent method which on like principles conducts to the same object.

From the simple, we will now proceed to more complex instances, and investigate expressions for the forces in those cases in which, besides the plane of the orbit, another, such as that of the ecliptio, is introduced.

Let the plane of the orbit be supposed to pass through ST, TL, and the second plane through ST, Tl; Ll being drawn from L perpendicularly to the last plane: let also the angle LTl be denoted by ϕ, its tangent by s; Tl by ρ, and the angle STl by ω. The resolution, then, of the forces into the three directions of Tl (the projected radius) of a line perpendicular to Tl, and of Ll perpendicular to the plane of STl, will, on the same principles as the former resolution (see p. 51.) be as follows,



$$\left.\begin{array}{l}\text{The difference } (D) \text{ of forces by which } L\\ \quad\text{and } T \text{ are urged towards } S\text{, in a direction}\\ \quad\text{parallel to } ST\end{array}\right\} = m'\left(\frac{r'}{y^3} - \frac{1}{r'^2}\right).$$

This, in the direction of Tl, $= m'\left(\frac{r'}{y^3} - \frac{1}{r'^2}\right)$ cos. ω,

and, in the direction perpendicular to $Tl = m'\left(\frac{r'}{y^3} - \frac{1}{r'^2}\right)$ sin. ω.

$$\left.\begin{array}{l}\text{Again, the force of } L \text{ to } S \text{ resolved}\\ \quad\text{first into the direction } LT\text{, and then}\\ \quad\text{into those of } Tl,\ Ll\text{, are re-}\\ \quad\text{spectively}\end{array}\right\} = \frac{m'}{y^2} \times \frac{r}{y} \times \frac{\rho}{r} = \frac{m'\rho}{y^3},$$

$$\text{and } = \frac{m'}{y^2} \times \frac{r}{y} \times \frac{Ll}{r} = \frac{m'\rho s}{y^3}.$$

We must now resolve the centripetal force $\left(\frac{\mu}{r^2}\right)$ into the directions Tl, Ll.

The resolved part of $\frac{\mu}{r^2}$, in the direction Tl, $= \frac{\mu}{r^2} \cdot \frac{\rho}{r} = \frac{\mu}{\rho^2} \cos.^3 \phi$,

in the direction Ll, $= \frac{\mu}{r^2} \times \frac{Ll}{r} = \frac{\mu}{\rho^2} \sin.\ \phi \cdot \cos.^2 \phi$.

Hence, collecting the several parts of the forces,

$$P = \frac{\mu}{\rho^2} \cos.^3 \phi + \frac{m'\rho}{y^3} + \left(\frac{m'}{r'^2} - \frac{m'r'}{y^3}\right) \cos \omega,$$

$$T = \left(\frac{m'}{r'^2} - \frac{m'r'}{y^3}\right) \sin.\ \omega,$$

$$S = \frac{\mu}{\rho^2} \sin.\ \phi \cdot \cos.^2 \phi + \frac{m'\rho}{y^3} \tan.\ \phi, \text{ or } = \frac{\mu}{\rho^2}\,\frac{s}{(1+s^2)^{\frac{3}{2}}} + \frac{m'\rho s}{y^3},$$

which expressions are the same as what Mayer has deduced in his *Theoria Lunæ*, p. 4.

Of these forces, the whole of T is a disturbing force, and of P and S, the parts that arise from the disturbing force, are

$$P - \frac{\mu}{\rho^2} \cos.^3 \phi = \frac{m'\rho}{y^3} + \left(\frac{m'}{r'^2} - \frac{m'r'}{y^3}\right) \cos \omega,$$

$$S - \frac{\mu}{\rho^2}\,\frac{s}{(1+s^2)^{\frac{3}{2}}} = \frac{m'\rho s}{y^3}.$$

These expressions for P, T, and S being substituted in the equations (4), (5), (6), of p. 10. and the *equations being solved,* every thing relating to the parallax, latitude and longitude, (see *Astronomy,* Chap. XXXIV.) of the disturbed body, or, to speak without reference to the elliptical theory, of a body agitated by the above forces, would become known. But, as we have already remarked (p. 59.), we must avail ourselves of every means to facilitate the solution; and, accordingly, if r' should be considerably greater than r, we ought to substitute, instead of $\frac{1}{y^3}$, its approximate expression; thus, substituting for $\frac{1}{y^3}$ as before (see p. 59.), and supposing, from the smallness of the inclination, r nearly to equal ρ,

$$P = \frac{\mu}{\rho^2}(1 + s^2)^{-\frac{3}{2}} - \frac{m'\rho}{2r'^3} - \frac{3m'\rho}{2r'^3}\cos. 2\omega,$$

$$T = \frac{3m'\rho}{2r'^3}\sin. 2\omega,$$

$$S = \frac{\mu}{\rho^2}\,\frac{s}{(1 + s^2)^{\frac{3}{2}}} + \frac{m'\rho s}{y^3}.$$

These expressions result by rejecting, (see p. 59.) from the expansion of $\frac{1}{y^3}$, the terms that involve higher powers of $\frac{r}{r'}$ than the cube. If we go a step farther, and retain the terms involving the next higher powers of $\frac{r}{r'}$, then

$$P = \frac{\mu}{\rho^2}(1 + s^2)^{-\frac{3}{2}} - \frac{m'\rho}{2r'^3}(1 + 3\cos. 2\omega) -$$

$$\frac{m'\rho^2}{8r'^4}\left(9\cos. \omega + 15\cos. 3\omega\right),$$

$$\text{and } T = \frac{3m'\rho}{2r'^3}\sin. 2\omega + \frac{m'\rho^2}{8r'^4}(3\sin. \omega + 15\sin. 3\omega).$$

(See *Mem. des Sçavans Etrangers,* Tom. VII. p. 14. also, Simpson's *Tracts,* p. 176.)

When, besides the plane of the body's orbit, another plane like that of the ecliptic is considered, the forces P, T, and S will consist of terms involving, besides constant quantities, ρ, v, and s; and, (yet this is a matter of mere analytical convenience,) they may be expressed by means of *partial* differential coefficients (see *Prin of Anal. Calc.* p. 79,) of a *function* of those quantities: thus, if v and v' be the *longitudes* of L and S, the angle STl, or $\omega = v - v'$,

$$\begin{aligned} \text{and } y &= \surd(Sl^2 + Ll^2) \\ &= \surd[r'^2 + \rho^2 - 2r'\rho \,.\, \cos.(v - v') + \rho^2 s^2]. \end{aligned}$$

Assume

$$R = m'\left(\frac{\rho \,.\, \cos.(v - v')}{r'^2} - \frac{1}{y}\right);$$

then, $$\frac{dR}{d\rho} = m'\left(\frac{\cos.(v - v')}{r'^2} + \frac{\rho - r'\cos.(v - v') + \rho s^2}{y^3}\right);$$

$$\frac{dR}{ds} = \frac{m'\rho^2 s}{y^3},$$

and $$\frac{dR}{dv} = -m'\left(\frac{\rho \sin.(v - v')}{r'^2} + \frac{r'\rho \sin.(v - v')}{y^3}\right);$$

$$\therefore \text{ (see p. 64.) } P = \frac{\mu}{\rho^2}\cos.^3\phi + \frac{dR}{d\rho} - \frac{s}{\rho} \,.\, \frac{dR}{ds},$$

$$T = \frac{1}{\rho} \,.\, \frac{dR}{dv},$$

$$S = \frac{\mu}{\rho^2} \,.\, \frac{s}{(1 + s^2)^{\frac{3}{2}}} + \frac{1}{\rho}\frac{dR}{ds},$$

and we may obtain expressions still more simple by assuming

$$Q = \frac{\mu}{r} - R = \frac{\mu}{\rho\surd(1 + s^2)} - R,$$

in which case,

$$\frac{dQ}{d\rho} = -\frac{\mu}{\rho^2\surd(1 + s^2)} - \frac{dR}{d\rho}, \text{ or } = -\frac{\mu u^2}{\surd(1 + s^2)} + u^2 . \frac{dR}{du}, \ \left(u = \frac{1}{\rho}\right),$$

$$\frac{dQ}{ds} = -\frac{\mu s}{\rho(1 + s^2)^{\frac{3}{2}}} - \frac{dR}{ds}, \text{ or } = -\frac{\mu us}{(1 + s^2)^{\frac{3}{2}}} - \frac{dR}{ds},$$

$$\frac{dQ}{dv} = -\frac{dR}{dv}.$$

Hence, since $\cos.^3\phi = \dfrac{1}{(1 + s^2)^{\frac{3}{2}}}$,

$$P = \frac{s}{\rho}\frac{dQ}{ds} - \frac{dQ}{d\rho}, \text{ or, } = s\,u\,.\frac{dQ}{ds} + u^2\frac{dQ}{du},$$

$$T = \frac{1}{\rho}\frac{dQ}{dv}, \text{ or, } = u\frac{dQ}{dv},$$

$$S = -\frac{1}{\rho}\frac{dQ}{ds}, \text{ or, } = -u\frac{dQ}{ds}\,*.$$

This mode of representing the forces, although simple and convenient, is not an obvious one, when P, T, S are to be expressed in terms of u, v and s. But it would be easily suggested, if the forces and equations were expressed in terms of the rectangular co-ordinates x, y, z. This is another advantage (see p. 40.) of these symmetrical equations, which we have not deduced before or elsewhere, for fear of interrupting the course of investigation.

Let, (see the figure of p. 63.) the co-ordinates of L reckoned from T, be x, y, z; of S, x', y', z';

$$\text{then } SL\ (\lambda) = \sqrt{[(x'-x)^2 + (y'-y)^2 + (z'-z)^2]},$$

and, on the same principles as before, (see p. 64.)

the force of S on L, in a direction parallel to x, $= \frac{m'}{\lambda^2} \times \frac{x'-x}{\lambda}$,

of S on T, in the same direction, $= \frac{m'}{r'^2} \times \frac{x'}{r'}$.

The difference then of these forces, on which the disturbing force depends $\Big\} = m'\left(\frac{x'-x}{\lambda^3} - \frac{x'}{r'^3}\right)$.

Hence, since the centripetal force in the direction parallel to x is $\frac{M+m}{r^2} \times \frac{x}{r}$, we have the whole force in that direction thus expressed (see p. 52.)

$$X = \frac{(M+m)\,x}{r^3} + \frac{m'(x-x')}{\lambda^3} + \frac{m'x'}{r'^3},$$

Y and Z will be similarly expressed: Hence, substituting their values in the equations of p. 8. there result

$\frac{d^2x}{dt^2}$

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* These expressions are the same as Laplace's *Mec. Cel.* Liv. II. p. 151. and Lib. VII. *Theorie de la Lune*, p. 181.

The advantage of thus expressing the forces consists in this: that, if Q should be expressed by part of an expanded function, in other words, by that part of the series which remains after

---

$$\frac{d^2x}{dt^2} + \frac{(M+m)x}{r^3} + m'\left(\frac{x-x'}{\lambda^3} + \frac{x'}{r'^3}\right) = 0,$$

$$\frac{d^2y}{dt^2} + \frac{(M+m)y}{r^3} + m'\left(\frac{y-y'}{\lambda^3} + \frac{y'}{r'^3}\right) = 0,$$

$$\frac{d^2z}{dt^2} + \frac{(M+m)z}{r^3} + m'\left(\frac{z-z'}{\lambda^3} + \frac{z'}{r'^3}\right) = 0.$$

Now it is obvious, that the third terms are partial differential coefficients of $\frac{1}{\lambda}$, and the fourth or last terms of $\frac{xx'+yy'+zz'}{r'^3}$; let, therefore,

$$\Omega = m'\left(\frac{xx'+yy'+zz'}{r'^3} - \frac{1}{\lambda}\right),$$

then

$$\frac{d^2x}{dt^2} + \frac{(M+m)x}{r^3} + \frac{d\Omega}{dx} = 0,$$

$$\frac{d^2y}{dt^2} + \frac{(M+m)y}{r^3} + \frac{d\Omega}{dy} = 0,$$

$$\frac{d^2z}{dt^2} + \frac{(M+m)z}{r^3} + \frac{d\Omega}{dz} = 0.$$

If we suppose, as in the Note to p. 52. L to be Jupiter and T Saturn, the three preceding equations will be those which we must use for determining Jupiter's motion: and, for determining Saturn's, we shall have, as it is plain from pp. 53, &c.

$$\frac{d^2x'}{dt^2} + \frac{(M+m')x'}{r'^3} + \frac{m}{m'}\cdot\frac{d\Omega}{dx'} = 0,$$

&c.

or, to render the two sets of equations more alike, we may thus express them

$$\frac{d^2x}{dt^2} + \frac{Mx}{r^3} + \frac{mx}{r^3} + \frac{m'x'}{r'^3} - m'\left(\frac{d\lambda}{dx}\right),$$

$$\frac{d^2x'}{dt^2} + \frac{Mx'}{r'^3} + \frac{m'x'}{r'^3} + \frac{mx}{r^3} - m\left(\frac{d\lambda}{dx'}\right).$$

&c.

See *Mem. Acad.* Paris, 1785. pp. 38.

The

the rejection of terms that are diminished beyond a certain minuteness, the quantities P, T and S might immediately be exhibited.

The values of P, T, S, being thus obtained, the place of the body may be determined, as we have already observed (p. 65.) by the solution of the equations (4), (5), (6). But these involve dt: this quantity, therefore, must be eliminated, as in the former case, (see p. 22.) by a process of the same principle and kind, but, by reason of the additional conditions, more operose.

The preceding equations differ from those which obtain (see p. 6.) in a system of two bodies, solely by the last terms $\frac{d\Omega}{dx}$, &c. This mode of representing the differential equations was first used by Lagrange, (see *Acad.* Berlin, 1776. p. 210. 1781. p. 214.) and it has been adopted by Laplace and other foreign mathematicians (see *Mec. Cel.* p. 254. also *Mem. Inst.* tom. IX. p. 12. &c.).

If, besides S, a fourth body (S') should disturb the motion of L round T, it would be necessary merely to add to the former value of Ω,

$$m''\left(\frac{xx''+yy''+zz''}{r''^3}-\frac{1}{\lambda'}\right),$$

m'', x'', y'', r'', &c. being its mass, co-ordinates, distance, &c. and then the preceding equations would obtain.

Since $r=\sqrt{(x^2+y^2+z^2)}$, $\frac{x}{r^3}=-\frac{d\frac{1}{r}}{dx}$; therefore, if we make $\phi=-\frac{\mu}{r}+\Omega$, there will result

$$\frac{d\phi}{dx}=\frac{\mu x}{r^3}+\frac{d\Omega}{dx},$$

and consequently, instead of the three preceding equations, we shall have

$$\frac{d^2x}{dt^2}+\frac{d\phi}{dx}=0,$$

$$\frac{d^2y}{dt^2}+\frac{d\phi}{dy}=0,$$

$$\frac{d^2z}{dt^2}+\frac{d\phi}{dz}=0.$$

Before we proceed to that operation (which will be one of the subjects of the ensuing Chapters), we wish to state the form under which the mathematicians, that succeeded Newton, considered the problem of the Moon's Orbit.

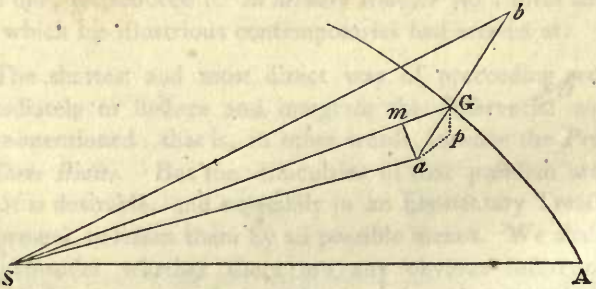
If we take no account of the disturbing force that acts perpendicularly to the plane of the body's orbit, then the forces acting on the body are two, P and T, and the integration of the equations would be the solution of a problem, in which it should be required to find the curve described by a body acted on by two forces, one in the direction of the radius, the other in a direction perpendicular to the radius. It was under the terms of this latter statement that Clairaut first proposed (see *Mem. Acad. des Sciences*, 1745, p. 342. and *Theorie de la Lune*, p. 3.) the question of the lunar orbit. The force in the direction of the radius in part arose from the mutual attraction of the Moon and Earth, and, in part, from the Sun's disturbing force. The other force that acted perpendicularly to the radius originated wholly from the Sun. Dalembert, (see *Recherches sur differens points*, &c.) although he derived his differential equations by a process different from that which Clairaut followed, yet, like him, reduced to a similar statement, the lunar theory. And Thomas Simpson, our countryman, notwithstanding some peculiarity in those Propositions which are preparatory to the main one, yet reduced it '*en demiere Analyse*' to a form similar to that which his illustrious contemporaries had arrived at.

The shortest and most direct way of proceeding would be immediately to deduce and integrate the differential equations above-mentioned: that is, in other words, to solve the *Problem of the Three Bodies*. But the difficulties of that problem are such, that it is desirable, and especially in an Elementary Treatise like the present, to lessen them by all possible means. We shall therefore consider whether there are any obvious inferences that present themselves on the introduction of a third disturbing body into the system of two bodies. The third body, as we have already seen, destroys the equable description of areas and the elliptical form of the orbit. The disturbing force of the Sun, for instance, prevents the Moon from describing an ellipse round the Earth, and the disturbing force of the Moon prevents the Earth from describing an ellipse round the Sun. Neither the Solar nor Lunar theories, therefore, can be exactly constructed according to Kepler's Laws. But, may not the point called the Centre of Gravity, which, in the doctrines of Equilibrium and Dynamics, possesses several curious properties, possess some here? May it not, notwithstanding the agency of disturbing forces on the bodies of the system, observe, either exactly, or nearly so, Kepler's Laws? The enquiries of mathematicians, in the rise of Physical Astronomy, would be naturally directed to this subject. Newton considered it in the first Propositions of the 11th Section. And, as it will soon appear, the laws of the motion of the centre of gravity, and the form of the curve described by it, belong to an investigation more difficult than the preceding, but less so than those that follow. We shall find ourselves, indeed, at a problem intermediate to the problems of two and of three bodies; and, although its solution is not essential and might be dispensed with, yet it is a *convenient* solution, will illustrate the subject matter of enquiry, and will furnish, in some instances, very easy methods of computing the effects of disturbing forces. For these reasons we shall briefly consider it in the following Chapter.

CHAP. VI.

The Motion of the Centre of Gravity of two or more Bodies not affected by their mutual Action: their Centre of Gravity attracted by a distant External body (the System revolving round it) by a Force nearly as the Inverse Square of the Distance: it describes therefore an Ellipse, nearly, round that Body. The Centre of Gravity of the Earth and Moon, the Centres of Gravity of Jupiter and his Satellites, of Saturn and his, all describe, very nearly, Ellipses round the Sun, and Areas proportional to the Times. Values of the Disturbing Forces that prevent the exact Description. The Moon's Menstrual Motion: Values of the Perturbations of her Parallax and Longitude by the Earth's Action: Value of the Menstrual Parallax.

If S represent a central attracting body, and a a body revolving round it, then, if the law of attraction should be inversely as the square of the distance, and no disturbing force should



operate, a (as it has been shewn in pp. 27, &c.) would revolve either in a circle, or in an ellipse round S, and would describe areas proportional to the time.

If b be introduced as a third, and as (see pp. 47. 49, &c.) a *disturbing* body, and if the distance (r) of a from S, be much greater than that (r') of b from a, and also if m', the mass of the disturbing body, be very small relatively to the masses (M and m) of S and a, then, as we have seen in p. 60. the two forces acting on a are

$$\frac{M+m+m'}{r^2}+\frac{m'\,.\,\cos.\,\omega}{r'^2}=P, \text{ and}$$

$$\frac{m'\sin.\,\omega}{r'^2}=T,$$

and, under the influence of these forces, the body a will no longer describe an ellipse round S, nor areas proportional to the time.

The preceding conditions and inferences will hold good, if S represent the Sun, a the Earth, and b the Moon: for in this instance

$$M = 1,$$

$$m = \frac{1}{329650},$$

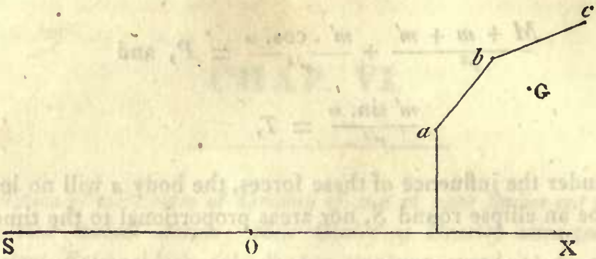
$$\frac{m'}{m} = \frac{1}{58\,.\,6},$$

$$\frac{r'}{r} = \frac{27\,.\,2}{10661}.$$

Take the point G such, that $\frac{aG}{bG}=\frac{m'}{m}$, then G is the centre of gravity of a and b: and the first point to be considered is this: how will the quiescence or motion of G be affected by the mutual action (supposing such action to be carried on according to the laws of attraction) of the bodies a and b? For, the curve described by G and the laws of its motion will depend partly on the motion which it has independently of S, and partly on the force by which it is urged to S. We will consider the first point, and instead of two bodies a and b, we will take more; so that the results may extend, beyond the case of the Earth and Moon, to those of Jupiter and Saturn and their systems of satellites.

For the sake of simplicity, suppose the bodies a, b, c, and whose masses are respectively m, m', m'', to be in the same plane: and let their respective rectangular co-ordinates, with reference to a point S and a line SO, be x, y, x', y', x'', y'',

x, x', x'', being measured in the direction of SO, and y, y', y'',



in directions perpendicular to SO:

Let

$$\lambda = ab = \sqrt{[(x'-x)^2 + (y'-y)^2]},$$
$$\lambda' = ac = \sqrt{[(x''-x)^2 + (y''-y)^2]},$$
$$\lambda'' = bc = \sqrt{[(x''-x')^2 + (y''-y')^2]};$$

then a is attracted towards b with a force $= \frac{m'}{ab^2}$, and in a direction parallel to SO, with a force

$$= \frac{m'}{ab^3} \times (x'-x) = \frac{m'\ (x'-x)}{\lambda^3} = m'' . \frac{1}{dx} d \left(\frac{1}{\lambda}\right) *.$$

Similarly, a is attracted by c, in a direction parallel to SO, by a force =

$$\frac{m''}{ac^3} . (x''-x) = m'' . \frac{x''-x}{\lambda'^3} = -m'' . \frac{1}{dx} . d \left(\frac{1}{\lambda'}\right),$$

and so on.

In like manner, the forces acting on a, in directions perpendicular to SO, are

$$-m' . \frac{1}{dy} . d \left(\frac{1}{\lambda}\right)$$

$$-m'' . \frac{1}{dy} . d \left(\frac{1}{\lambda'}\right).$$

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* $\frac{1}{dx} d \left(\frac{1}{\lambda}\right)$ is meant to denote the *partial* differential coefficient of $\frac{1}{\lambda}$: the differential being taken relatively to x.

Now, see pp. 5, &c. if X represent the sum of forces acting parallel to x, the equation involving x is

$$\frac{d^2 x}{d t^2} + X = 0.$$

Hence, for determining the effect of the attractions on a that are parallel to SO, we have

$$\frac{d^2 x}{d t^2} - m' \cdot \frac{1}{dx} d \left(\frac{1}{\lambda}\right) - m'' \cdot \frac{1}{dx} d \left(\frac{1}{\lambda'}\right) - \&c. = 0,$$

or,

$$\frac{d^2 x}{d t^2} - \frac{1}{m} \cdot \frac{1}{d x} \cdot d \left(\frac{m m'}{\lambda} + \frac{m m''}{\lambda'} + \&c.\right) = 0,$$

and, when the direction is parallel to the ordinates y, y', &c. the corresponding equation is

$$\frac{d^2 y}{d t^2} - \frac{1}{m} \cdot \frac{1}{d y} d \left(\frac{m m'}{\lambda} + \frac{m m''}{\lambda'} + \&c.\right) = 0.$$

Similar equations obtain for the attractions on b and c, which evidently will be

$$\frac{d^2 x'}{d t^2} - \frac{1}{m'} \cdot \frac{1}{d x'} d \left(\frac{m' m}{\lambda} + \frac{m'' m'}{\lambda''} + \&c.\right) = 0,$$

$$\frac{d^2 y'}{d t^2} - \frac{1}{m'} \cdot \frac{1}{d y'} d \left(\frac{m' m}{\lambda} + \frac{m'' m'}{\lambda''} + \&c.\right) = 0.$$

$$\frac{d^2 x''}{d t^2} - \frac{1}{m''} \cdot \frac{1}{d x''} d \left(\frac{m'' m}{\lambda'} + \frac{m'' m'}{\lambda''} + \&c.\right) = 0.$$

$$\frac{d^2 y''}{d t^2} - \frac{1}{m''} \cdot \frac{1}{d y''} d \left(\frac{m'' m}{\lambda'} + \frac{m'' m'}{\lambda''} + \&c.\right) = 0.$$

Add all the equations involving $d^2 x$, $d^2 x'$, &c. together, then, since

$$\frac{1}{d x} d \left(\frac{m m'}{\lambda}\right) = - \frac{1}{d x'} \cdot d \left(\frac{m m'}{\lambda}\right)$$

$$\frac{1}{d x} d \left(\frac{m m''}{\lambda'}\right) = - \frac{1}{d x''} \cdot d \left(\frac{m m''}{\lambda'}\right),$$

&c. &c.

there results

$$m \cdot \frac{d^2 x}{d t^2} + m' \cdot \frac{d^2 x'}{d t^2} + m'' \cdot \frac{d^2 x''}{d t^2} + \text{\&c.} = 0 \ldots\ldots (a),$$

and similarly,

$$m \cdot \frac{d^2 y}{d t^2} + m' \cdot \frac{d^2 y'}{d t^2} + m'' \cdot \frac{d^2 y''}{d t^2} + \text{\&c.} = 0 \ldots\ldots (b),$$

now, if X, Y be the co-ordinates of the centre of gravity, we have, by the property of that centre *,

$$(m + m' + m'' + \text{\&c.})\, X = m x + m' x' + m'' x'' + \text{\&c.}$$
$$(m + m' + m'' + \text{\&c.})\, Y = m y + m' y' + m'' y'' + \text{\&c.}$$

$$\therefore\ (m + m' + m'' + \text{\&c.})\, d^2 X = m d^2 x + m' d^2 x' + m'' d^2 x'' + \text{\&c.}$$
$$= 0, \text{ (by the equation } [a]),$$

and, similarly,

$$(m + m' + m'' + \text{\&c.})\, d^2 Y = 0.$$

Hence, to determine the motion of the point G, we have

$$\frac{d^2 X}{d t^2} = 0,$$

$$\frac{d^2 Y}{d t^2} = 0.$$

There are no *forces* then (see p. 5.), arising from the mutual action of a, b, c, &c. that solicit G : but, a point, or body, unsolicited by any forces is either at rest or moves uniformly in a right line: and this is the property of G the centre of gravity of the bodies a, b, c, &c.†

Thus the first point is established : the second point respects the *law* of the force by which G is attracted to S: if it should appear to be according to the inverse square of the distance, then there would follow, as an immediate and certain consequence, the description of an ellipse by the point G round the centre S.

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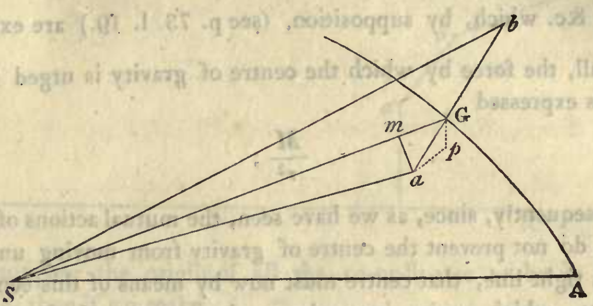
* See Wood's *Mechanics*, Prop. XXXVIII. Bridge's, p. 133. ed. 1.

† If more points, and other co-ordinates z, z', &c. had been used, the processes and results would have been exactly similar.

Let us resume (for the sake of simplicity) the first figure. The attraction of a to S, in a direction parallel to SG, is



$$= \frac{M \,.\, SG}{S a^3},$$

$$= \frac{M \,.\, SG}{(SG - aG \,.\, \cos.\, SGa)^3},$$

$$= \frac{M \,.\, r}{(r - p \,.\, \cos.\, \theta)^3} \text{ (making } SG = r,\ aG = p,\ bG = q,\ SGa = \theta)$$

$$= \frac{M}{r^2} + 3M\frac{p}{r^3} \cos.\, \theta + \frac{6M \,.\, p^2}{r^4} \cos.^2 \theta + \&c.$$

In like manner, the attraction of b to S, in a direction parallel to SG,

$$= \frac{M}{r^2} - \frac{3Mq \,.\, \cos.\, \theta}{r^3} + \frac{6Mq^2}{r^4} \cos.^2 \theta - \&c.$$

Now the force by which the centre of gravity is urged towards S being

$$\left(\frac{M \,.\, SG}{S a^3} \times m + \frac{M \,.\, SG}{S b^3} \times m'\right) \frac{1}{m + m'};$$

must, by substituting the preceding values of $\frac{M.SG}{S a^3}$, and $\frac{M.SG}{S b^3}$, equal

$$\frac{M}{r^2} + \left(\frac{3M}{r^3} \cos.\, \theta \,.\, (mp - m'q)\right) \frac{1}{m + m'} + \&c.$$

but, by the property of the centre of gravity,

$$mp - m'q = 0;$$

if we neglect therefore, the terms that involve $\frac{p^2}{r^4}$, $\frac{q^2}{r^4}$, $\frac{p^3}{r^5}$, &c. which, by supposition, (see p. 73. l. 19.) are extremely small, the force by which the centre of gravity is urged to S is thus expressed

$$\frac{M}{r^2};$$

consequently, since, as we have seen, the mutual actions of a, b, c, &c. do not prevent the centre of gravity from moving uniformly in a right line, that centre must now by means of this centripetal force, which varies inversely as the square of the distance, describe an ellipse round S.

There will be no alteration made in this result, if we consider the action of a and b on S; for the sum of their actions in a direction parallel to SG, will be

$$\frac{m \,.\, SG}{Sa^3} + \frac{m' \,.\, SG}{Sb^3}$$

$$= (m + m')\frac{1}{SG^2},$$

if, as before, we reject the small terms that involve $\frac{p^2}{r^4}$, &c.

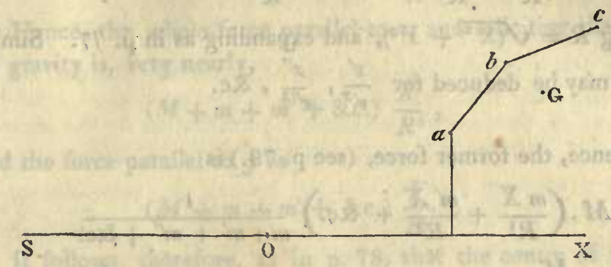
If we suppose S to be immoveable, and attribute to it (see p. 43.) this latter force $\frac{m + m'}{SG^2}$, we shall have the whole force by which G is urged towards S equal to

$$\frac{M + m + m'}{SG^2}.$$

The preceding demonstration may be extended to any number of bodies a, b, c, &c. forming a system, by supposing a and b collected in G, and then by taking a new centre of gravity of c and of $a + b$ collected in G.

But, it is easy to obtain at once a general demonstration by the aid of such symbols, equations and processes as were used in pp. 74, &c.

Let X, Y be the co-ordinates of the centre of gravity G, and



let us suppose the origin of all the co-ordinates to be in the centre S, then if we make

$$x = X \mp x_{,},$$
$$x' = X \mp x'_{,},$$
$$\&c.$$

$x_{,}$, $x'_{,}$, &c. will be the co-ordinates of a, b, c, relatively to the point G. Now, since it has appeared that the mutual actions of a, b, c, &c. have no influence on the motion of the centre (G) of gravity, it is not necessary, in investigating its motion, to attend to any other force than that of the external body S: now the action of S on a, in a direction parallel to x, is

$$\frac{Mx}{r^3}, \ [r = \sqrt{(x^2 + y^2)}],$$

on b, it is

$$\frac{Mx'}{r'^3},$$

on c, it is

$$\frac{Mx''}{r''^3};$$

consequently, the force soliciting the centre of gravity in a direction parallel to x, is

$$\left(\frac{Mx}{r^3} \times m + \frac{Mx'}{r'^3} \times m' + \&c.\right) \frac{1}{m+m'+m''}.$$

$$\text{But, } \frac{x}{r^3} = \frac{X \mp x_{,}}{[(X \mp x_{,})^2 + (Y \mp y_{,})^2]^{\frac{3}{2}}}$$

$$= \frac{X}{R^3} \mp \frac{x_{,}}{R^3} \pm \frac{3\,X\,(X x_{,} + Y y_{,} + \&c.)}{R^5},$$

making $R = \sqrt{(X^2 + Y^2)}$, and expanding as in p. 77. Similar forms may be deduced for $\frac{x'}{r'^3}$, $\frac{x''}{r''^3}$, &c.

Hence, the former force, (see p. 79.) is

$$M.\left(\frac{m\,X}{R^3} + \frac{m'\,X}{R^3} + \&c.\right)\frac{1}{m + m' + m'' + \&c.}$$

$$-\frac{M}{R^3}\,(m\,x + m'\,x' + m''\,x'' + \&c.)\,\frac{1}{m + m' + m'' + \&c.}$$

$$\pm\,\frac{3\,M X}{R^5}\,(m\,X\,x_{,} + m\,Y y_{,} + \&c.)\,\frac{1}{m + m' + m'' + \&c.}.$$

Now, by the property of the centre of gravity,

$$m\,x + m'\,x' + \&c. = 0.$$

Hence, if we neglect quantities involving $\frac{x_{,}}{R^5}$, &c., the force which solicits the centre of gravity, in a direction parallel to x, is

$$\frac{M\,.\,X}{R^3}.$$

Similarly, the force which acts on the same centre, in a direction parallel to y, is

$$\frac{M\,.\,Y}{R^3}.$$

But these are the same forces that would solicit the whole system of bodies m, m', &c. collected in the centre of gravity.

If we consider the actions of m, m', m'', &c. on S, their sum parallel to x will be

$$\frac{m\,x}{r^3} + \frac{m'\,x'}{r'^3} + \frac{m''\,x''}{r''^3} + \&c.$$

which, by the preceding process, (see l. 1.) may very nearly, (that is, by rejecting very small quantities such as $\frac{x_{,}}{R^5}$, &c.), be expressed by

$$(m + m' + m'' + \&c.)\,\frac{X}{R^3}.$$

Hence, the whole force parallel to x, and soliciting the centre of gravity is, very nearly,

$$(M + m + m' + \&c.)\,\frac{X}{R^3},$$

and the force parallel to y is

$$(M + m + m' + \&c.)\,\frac{Y}{R^3}\,{}^*,$$

It follows, therefore, as in p. 78. that the centre of gravity must move in the same way as if all the bodies m, m', m'', &c. were collected in it; being deflected, therefore, from its uniform rectilinear course by a force varying inversely as the square of the distance, it must describe an ellipse.

The centre of gravity then of the Moon and Earth describes very nearly an ellipse round the Sun, and the centres of gravity of Jupiter and his satellites, of Saturn and his satellites, describe very nearly, but not accurately, ellipses round the Sun: the first of these results is what Newton asserts to be true †.

The centre of gravity does not describe *accurately* an ellipse round the Sun: for, as we have seen in p. 77. the force by which that centre is urged is not exactly

$$\frac{M + m + m'}{r^2};$$

but

$$\frac{M+m+m'}{r^2} + \left(\frac{6Mm.p^2}{r^4}.\cos.^2\theta + \frac{6Mm'q^2}{r^4}\cos.^2\theta\right)\frac{1}{m+m'} + \&c.$$

This force, however, which urges G follows the law of the inverse square of the distance much more nearly than that which urges the body a; for, (see p. 73.) this latter force is equal to

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* This demonstration will, on consideration, be found to be exactly the same in principle as the preceding one of p. 77.

† Commune centrum gravitatis terræ et lunæ, ellipsin circum solem in umbilico positum percurrit, et radio ad solem ducto areas in eadem temporibus proportionales describit. Terra vero circum hoc centrum communæ motu menstruo revolvet, Prop. XIII. Lib. 3.

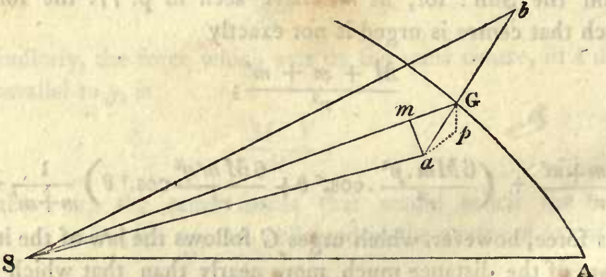
$$\frac{M + m + m'}{r^2} + \frac{m' . \cos. \omega}{r'^2},$$

not exactly so, but by neglecting (see p. 60.) terms involving $\frac{m' r'}{r^3}$, &c.: now r' expresses the distance of a from b: for instance, in the case of the Earth's perturbation by the action of the Moon, r' is the radius of the Moon's orbit, which corresponds in the preceding case to $p + q$; hence, the approximate expression for the force urging a is obtained by rejecting larger quantities than those which are rejected in obtaining

$$\frac{M + m + m' + \&c.}{SG^2},$$

which is the approximate expression for the force soliciting the centre of gravity, and is exactly after the law of the inverse square of the distance; which the other is not.

The centre of gravity, then, of the Earth and Moon, (if we still illustrate our reasonings by that case,) describes very nearly an ellipse round the Sun: it describes also, very nearly, areas proportional to the time. The small deviation from this last Law of Kepler, is of the same degree as the deviation from the former, since it depends on terms of the same magnitude, or degree of minuteness: for, the force by which, from the attractions on a and b, G is urged in the direction $a\,b$, is



$$\left(\frac{M m . a G}{S a^3} - \frac{M m' . b G}{S b^2}\right) \frac{1}{m + m'},$$

which, by the forms of p. 77. is reduced to

$$\frac{3 M . \cos. \theta}{r^4 . (m + m')} . (m p^2 + m' q^2) - \&c.$$

This force, which acts obliquely to SG may be resolved into two others, one in the direction of SG *, the other in a direction perpendicular to SG: and this last will, as it is plain, disturb the equable description of areas; but, in the case of the Moon and Earth, or in the cases of Jupiter, Saturn and their satellites, will, from the great minuteness of $\frac{p^2}{r^4}$, $\frac{q^2}{r^4}$ very slightly disturb it.

The force (see p. 60.) which acts perpendicularly to the line joining S and a is

$$\frac{m' \sin. \omega}{r'^2}.$$

The force, therefore, which prevents the body a, or in the instance we are using, which prevents the Earth from describing equal areas in equal times, round the Sun, is much more considerable than that similar portion of the disturbing force which acts on the centre of gravity.

In the expression for the force acting on the centre of gravity (see pp. 77, &c.) all the terms, saving the first, are exponents of the disturbing force. The first and most considerable of those terms is equal to

$$\frac{3M}{r^4}(m p^2 + m' q^2)\ (1 + \cos. 2\theta) \times \frac{1}{m + m'},$$

$$\text{or, } 3M \,.\, \frac{m'}{m} \,.\, \frac{1}{\left(1 + \frac{m'}{m}\right)^2} \,.\, \frac{r'^2}{r^4}\ (1 + \cos. 2\theta)$$

Now, since, during the Moon's revolution, the $\cos. 2\theta$ passes through all the degrees of magnitude that are between 1 and − 1, it is plain that the disturbing force will be sometimes additive and sometimes subtractive, and equally so: consequently, if we suppose an ellipse to be described by the centre of gravity round the Sun, and which would indeed be described were the sole force urging that centre,

$$\frac{M + m + m'}{SG^2};$$

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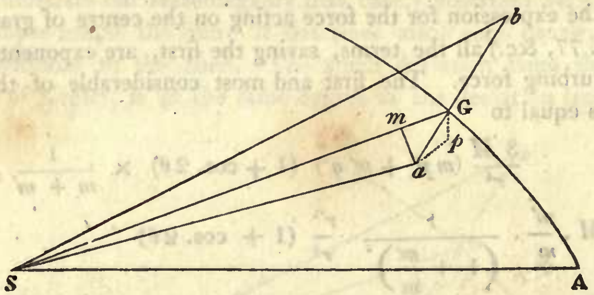
* This resolved part (which, however, is very small) must be added, in order to complete it, to the force of p. 81. l. 22.

then the more true path of the centre of gravity will be a curve sometimes within and sometimes without such ellipse, and repeatedly intersecting it. It will be, as Dalembert has said,* a species of *Epicycle*, and the curve described by the Earth and by the Moon round the centre of gravity, a species of *Eccentric*.

The deviations, however, of the Earth, in its course, from the ellipse described by the centre of gravity are very inconsiderable. They are amongst the least of all the perturbations that the Earth suffers from the action of the planets. For, we may view the deviations (such as are the present subjects of discussion) as entirely originating from Lunar disturbance. Thus, the Earth would describe an ellipse round the Sun, if the Moon were supposed to be abstracted; if, therefore, we take, for that ellipse, the ellipse described by the centre of gravity, the Moon is the cause why the Earth is not found in that curve. In the figure which we have used, suppose G to be the Earth's place



computed, or found by the strictly elliptical theory: then, if the Earth be at a, and $a\,m$ be drawn perpendicular to SG, $m\,G$ is the *perturbation* of the Earth's distance caused by the Moon, and

$$= a\,G \,.\, \cos.\ \angle\, m\,G\,a$$

$$= a\,b \times \frac{m'}{m + m'} \times \cos.\ \angle\, m\,G\,a$$

$$= -\frac{r'\,m'}{m + m'} \times \cos.\,(v - v').$$

supposing v and v' to be the longitudes of the Sun and Moon seen from G; or, if we suppose v to be the Earth's longitude seen from the centre of the Sun, then we must write $180 + v$ instead of v, and accordingly, we shall have,

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* Recherches sur differens points importans du Systême du Monde, Tom. II. p. 20.

$$dr\,(= -mG) = -\frac{m'}{m+m'}\,r'.\cos.(v'-v).$$

In like manner, the perturbation in longitude, or,

$$dv = \frac{am}{Sa} = \frac{aG.\sin.mGa}{Sa} = -\frac{m'}{m+m'}.\frac{r'}{r}\sin.(v'-v).$$

If from the values of m, m', r, r', given in p. 73. we arithmetically exhibit the coefficients of the preceding inequalities, we shall have,

$$dr = -.0000428\cos.(v'-v),$$

and

$$dv = -8''.8.\sin.(v'-v).$$

We see then how very small the perturbations caused by the Moon are: the greatest error in longitude which arises from this *menstrual** motion does not amount to nine seconds.

It may, perhaps, be worth the while to consider, a little more minutely, the effect of this menstrual motion.

Suppose † $bBb'B$ the Moon's orbit: when the Moon is at B, and the Earth at e, that is, at the time of new Moon, the Sun S is seen in the same part of the heavens, whether it be viewed from e or G; but if viewed from G it must be seen in its computed place; in this case, therefore, the observed and computed places agree, or there is no apparent irregularity in the Sun's motion.

The same holds, at the time of full Moon, when the Moon is at B', and the Earth at e'.

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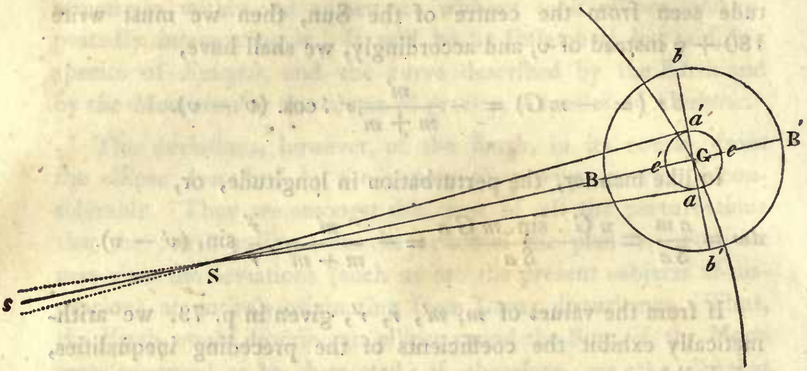
* Terra vero ~~arca~~ circa hoc centrum commune motu *menstruo* revolvetur. Newton, Prop. XIII. Lib. 3.

† See Figure in next page.

At the time of the Moon's first quarter, that is, when the



Moon is at b' and the Earth at a', a spectator at a' sees the Sun in the direction of a line drawn from a' through S, in a direction therefore to the left hand or to the eastward of his computed place, seen by a spectator placed at G. In the beginning of the Moon's last quarter, or when the Moon is in quadrature at b, a spectator at a would see the Sun in the direction of a line drawn from a through S: therefore, to the right or the westward of that place, where the Sun would appear to be to a spectator at G.

What has preceded, regards the perturbation in the Earth's motion caused by the Moon. But any one of Jupiter's satellites stands in the same relation to Jupiter that the Moon does to the Earth. The formula, therefore, of p. 85. may, *mutatis mutandis*, be used for determining Jupiter's perturbation caused by any one of his satellites, and consequently, to the perturbation caused by all: for this last will be merely the sum of the individual perturbations: the whole *inequality* therefore of the radius vector will be

$$-\frac{m'}{M+m'} \cdot r' \cdot \cos.\,(v-v')$$

$$-\frac{m''}{M+m''} \cdot r'' \cdot \cos.\,(v-v'')$$

$$-\frac{m'''}{M+m'''} \cdot r''' \cdot \cos.\,(v-v'''),\ \&c.$$

Supposing M to be the mass of Jupiter, m', m'', &c. the masses of his satellites, r', r'', &c. their radii, and v', v'', &c. their longitudes; and the inequality, or perturbation of his motion in longitude will be

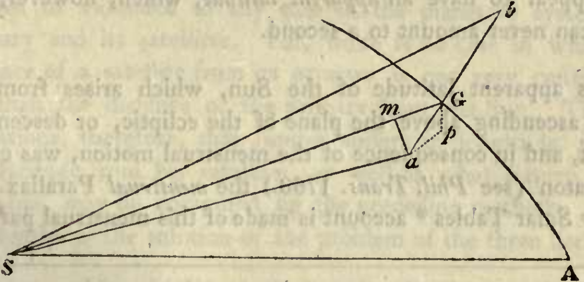
$$-\frac{m'}{M+m'}\cdot\frac{r'}{R}\,\text{sin.}\,(v - v)$$

$$-\frac{m''}{M+m_{,,}}\cdot\frac{r''}{R}\,\text{sin.}\,(v'' - v)$$

$$-\frac{m'''}{M+m'''}\cdot\frac{r'''}{R}\,\text{sin.}\,(v''' - v),\ \&\text{c.}$$

R being the radius of Jupiter's orbit.

We have, for the sake of simplicity, supposed the plane of the orbit in which the centre of gravity moves, to be coincident with the plane of the orbit in which the disturbing satellite moves round the disturbed primary. This supposition, although it very slightly diminishes (if we look to its effect in computation) the quantities of the perturbations in parallax and longitude, entirely suppresses the perturbation in latitude. In order to restore it, suppose $a\,G$, the line joining the Earth, (if that be the primary), and G, the centre of gravity of the Earth and Moon,



to be inclined, as it really is, to the plane of G's orbit round S: conceive, $a\,p$ to be perpendicular to that latter plane; then the perturbation in latitude will be

$$\frac{a\,p}{S\,a},$$

and if s be the tangent of the Moon's latitude, we shall have, nearly,

$$ap = r's \cdot \frac{m'}{m + m'},$$

and, consequently, the perturbation in latitude will equal

$$\frac{m'}{m + m'} \cdot \frac{r's}{r};$$

$$\text{or, } \frac{m'}{m + m'} \cdot \frac{r'}{r} \cdot \tan.\ \phi \cdot \sin.\ (v - \theta),$$

ϕ being the inclination of the plane of the Moon's orbit, and $v - \theta$ the Moon's distance from her node. Arithmetically expressed, the perturbation in latitude will be

$$0''.7847 \cdot \sin.\ (v - \theta).$$

In consequence of the perturbation in longitude, or, as it may be stated, in consequence of the Earth's motion round the centre of gravity of the Earth and Moon, the Sun will appear sometimes before and sometimes behind his *elliptically* computed place; but the deviation will never reach nine seconds: and, in consequence of the perturbation in latitude, the Sun will sometimes appear to have an *apparent latitude*, which, however, (see l. 10.) can never amount to a second.

This apparent latitude of the Sun, which arises from the Earth's ascending above the plane of the ecliptic, or descending below it, and in consequence of the menstrual motion, was called by Smeaton (see *Phil. Trans.* 1768.) the *menstrual* Parallax. In the new Solar Tables * account is made of this menstrual parallax in computing the *Sun's latitude.* The other perturbations that cause this inequality and latitude, arise from the actions of Venus and Jupiter.

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* See Delambre's *Tables du Soleil,* Table XV., and Vince's *Astronomy,* Vol. III. Table XXVI. This perturbation in latitude will affect the determination of the obliquity from the observed altitudes of the Sun near the solstices: it will also affect the point of time at which the Sun enters the equinox, that time being determined by observations of the Sun made near the equinoxes.

It has been already remarked, that the computation of the apparent deviations of the Sun's place by considering the Earth's menstrual motion, is only another mode of viewing and computing the effect of the Moon's perturbation. From the peculiarity of its conditions, it happens to be a very convenient mode, and it enables us to dispense with the formal solution of the problem of the three bodies. But, generally considered, and on principles of Philosophical Classification, it belongs to that latter problem, and ought to be included in the resulting formula which generally assigns the perturbation of any inferior planet.

The computation, therefore, of the Earth's perturbation caused by the Moon belongs to the problem of the three bodies: but it is the most simple and easy case of that problem, a circumstance to be presumed, since the problem itself is most difficult, and the same results, which its solution would afford, have, as we have seen, been obtained, with little trouble, by the preceding processes.

The preceding processes and reasonings have been principally applied to the Earth and Moon: but, it is plain they may be applied or extended to any case in the planetary system of a primary and its satellites. For, there is no case in which the distance of a satellite from its primary is not very considerably less than the distance of the primary from the Sun. The perturbations, therefore, that Jupiter suffers in longitude, parallax and latitude from his satellites, as well as what Saturn suffers from his, may be computed by the preceding methods, and antecedently to the solution of the problem of the three bodies.

That problem is the principal one in the theory of planetary perturbation: as soon as a *third* body is introduced into the mathematical system, Kepler's beautiful Laws are violated. The principal planets no longer describe ellipses round the Sun, placed in one of the foci: and this is true, whether the planets have satellites or not. The reasonings in this Chapter, indeed, establish this point only in the former case. The centres of gravity in their motions deviate a little from elliptical paths, the primary planets deviate more.

Since the centre of gravity of a primary planet and its satellites does not describe an ellipse, and the primary itself moves in an irregular curve with epicycloidal motions, an enquiry naturally suggests itself concerning the planet's actual recess from the Sun, or approach to it. Will the planet in its second revolution exactly regain, or *re-enter* into that irregular curve which it described in its first? Is there any law which, after a lapse of time, and under like configurations, preserves it at the same invariable distance from the Sun? These are questions extremely curious and important, since on them depends the permanence of the planetary system. Observation, indeed, assures us of the invariability of the mean motions of the planets, and thence we infer the invariability of their mean distances. But we look to Physical Astronomy for an assurance and proof that the mean distances must remain constant; a result by no means obvious, nor to be inferred from what has preceded, but rather, (if we may so express ourselves) a result *not to be presumed*, seeing the varied irregularities of force to which the planets are continually subjected. This enquiry will be resumed in another part of the Work.

At the end of the last Chapter, we stated that the investigations of the present would connect the problem of two, with the *Problem of the three Bodies;* and some results belonging to that latter have been obtained independently of it, and by peculiar methods; namely, the perturbations of the primary planets by their secondaries. But we in vain should attempt other cases by like methods. The same system of the Sun, Earth and Moon, affords us the most easy and the most difficult case of the problem of the three bodies: the former is the Earth revolving round the Sun, and disturbed by the Moon: the latter, the Moon revolving round the Earth and disturbed by the Sun. No two cases, in principle and in a general view are more alike, or more unlike in practical detail and difficulty.

In the succeeding Chapters we will return to the main investigation: for, the present is somewhat in the nature of a digression. It certainly contains no methods which lessen the real difficulties that must subsequently be encountered, namely, the difficulties of integrating the differential equations of motion, (see p. 70.). Their perfect integration would be a complete solution of the problem of the three bodies, and that solution would, in fact, comprehend the Lunar and Planetary theories; at least, all their essential parts. But hitherto nothing beyond an *approximate* solution has been drawn from the resources of analytical science; nor indeed seems likely to be drawn, if we attend to the expectations of those who have chiefly busied themselves in these enquiries.

CHAP. VII.

Elimination of dt *from the Differential Equations. The Three Equations that belong to the Theory of the Moon, and the Problem of the Three Bodies. The Approximate Integration of these Equations by the Method called the Variation of the Parameters. Application of that Method to particular Instances.*

THE investigations of this Chapter are, almost entirely, mathematical. They are connected, however, with Physical Astronomy, inasmuch as they lead to methods by which, on its principles, the celestial phenomena are explained.

The three equations on which the determination of the body's place depends, are

$$d^2\rho - \rho\, dv^2 + P\, dt^2 = 0 \ldots\ldots\ldots\ldots [4],$$

$$2\, d\rho\, dv + \rho\, d^2 v \pm T . dt^2 = 0 \ldots\ldots\ldots\ldots [5],$$

$$d^2(\rho s) + S . dt^2 \qquad = 0 \ldots\ldots\ldots\ldots [6],$$

and, dt must be eliminated by a process similar to the one of p. 22. in which $T = 0$.

First multiply the equation (5) by ρ and integrate it, then

$$\rho^2 . dv = h dt \mp dt \int \rho . T dt;$$

$$\therefore T\rho^3 dv = h . T\rho dt \mp T\rho dt \int T\rho . dt,$$

and again, integrating,

$$\int T\rho^3 dv = h \int T\rho . dt \mp \tfrac{1}{2} \left(\int T\rho . dt\right)^2.$$

Hence, finding $\int T\rho\, dt$ by the solution of a quadratic equation, and taking the lower sign that occurs in the preceding equation,

$$\int T\rho\, dt = -h + \sqrt{[h^2 + 2\int T\rho^3 dv]};$$

$$\text{whence, } T\rho\, dt = \frac{T\rho^3 dv}{\sqrt{[h^2 + 2\int T\rho^3 dv]}},$$

$$\text{and } dt = \frac{\rho^2 dv}{\sqrt{[h^2 + 2\int T\rho^3 dv]}};$$

or, in terms of u,

$$= \frac{d\,v}{h\,u^2 \sqrt{\left(1 + \frac{2}{h^2} \int \frac{T\,dv}{u^3}\right)}} \text{*}.$$

From this value of $d\,t$ we must, as before, (p. 22.) transform the equation (4), in which dt is constant, into another equation in which $d\,v$ shall be constant. This may be effected by the following method:

Since, $\frac{1}{d\,t^2} = \frac{h^2 + 2\int T\rho^3\,d\,v}{\rho^4\,d\,v^2}$,

by taking the differential of each side of the equation, there results

$$\frac{d^2\,t}{d\,t^3} = \frac{2\,d\,\rho}{dv^2} \times \frac{h^2 + 2\int T\rho^3\,d\,v}{\rho^5} - \frac{T}{\rho\,d\,v}.$$

But see p. 22. l. 14.

$$\frac{1}{d\,t}\left(\frac{d^2\,\rho}{d\,t} - \frac{d^2\,t}{d\,t^2}\,d\rho\right) - \rho\,\frac{d\,v^2}{d\,t^2} + P = 0.$$

Substitute in this the preceding values of $d\,t$ and $d^2\,t$, and there will result

$$\left[\left(\frac{d^2\,\rho}{\rho^2} - \frac{2\,d\,\rho^2}{\rho^3}\right)\frac{1}{dv^2} - \frac{1}{\rho}\right] \times \left(\frac{h^2 + 2\int T\rho^3\,dv}{\rho^2}\right) + P + \frac{T}{dv} \times \frac{d\,\rho}{\rho} = 0;$$

or, since $\frac{d^2\,\rho}{\rho^2} - \frac{2\,d\,\rho^2}{\rho^3} = d\left(\frac{d\,\rho}{\rho^2}\right) = d\,(-\,d\,u) = -\,d^2\,u$,

and $\frac{d\rho}{\rho} = \frac{d\,\rho}{\rho^2} \times \rho = -\,d\,u \times \frac{1}{u}$,

$$\frac{d^2\,u}{d\,v^2} + u + \frac{T\,\frac{d\,u}{dv} - P\,u}{u^3\left(h^2 + 2\int \frac{T}{u^3}\,d\,v\right)} = 0.$$

This is the form of the equation as it was first exhibited by Clairaut and Dalembert; and almost all subsequent authors have adopted the same. In its integration consists the solution of the

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* See Clairaut *Theorie de la Lune,* ed. 2. p. 4. Dalembert, *Recherches,* &c. tom. I. p. 16. Laplace, *Mec. Celeste.* Liv. VII. p. 181.

Problem of the Three Bodies. For, the equation (6) of p. 92. is easily reducible to the same form, and reducible by artifices the same as those that have already been used (see p. 22.): thus

$$\frac{d^2(\rho s)}{dt^2} + S = 0, \text{ or, } \frac{1}{dt}\, d\left(\frac{d.\rho s}{dt}\right) + S = 0,$$

and, making dt variable,

$$\frac{1}{dt}\;\frac{d^2(\rho s)}{dt} - \frac{1}{dt}.d.(\rho s)\frac{d^2 t}{dt^2} + S = 0.$$

Substitute in the preceding equation the values of dt and of $d^2 t$ as given in p. 93, and then

$$\frac{1}{\rho}\;\frac{d^2(s\rho)}{dv^2} - \frac{1}{\rho^2}\;\frac{d(s\rho).2d\rho}{dv^2} + \frac{T\rho^2\dfrac{d(s\rho)}{dv} + S\rho^3}{h^2 + 2\int T\rho^3 dv} = 0,$$

or, by reduction,

$$\frac{d^2 s}{dv^2} + \frac{s}{\rho}\left(\frac{d^2\rho}{dv^2} - \frac{2d\rho^2}{\rho.dv^2}\right) + \frac{T\rho^2\dfrac{d(s\rho)}{dv} + S\rho^3}{h^2 + 2\int T\rho^3 dv} = 0.$$

But, $d^2\rho - \dfrac{2d\rho^2}{\rho} = \rho^2\left(\dfrac{d^2\rho}{\rho^2} - \dfrac{2d\rho^2}{\rho^3}\right) = \rho^2 d\left(\dfrac{d\rho}{\rho^2}\right)$ $= -\rho^2 d(du) = -\dfrac{1}{u^2}d^2 u.$ Hence, the preceding equation becomes

$$\frac{d^2 s}{dv^2} - \frac{s}{u}.\frac{d^2 u}{dv^2} + \left[T\frac{d\left(\dfrac{s}{u}\right)}{dv} + \frac{S}{u}\right] \times \frac{1}{u^2\left(h^2 + 2\int\dfrac{T}{u^3}dv\right)} = 0,$$

which, by means of substituting for $d^2 u$ its value as assigned in p. 93. assumes this form

$$\frac{d^2 s}{dv^2} + s + \frac{T\dfrac{ds}{dv} + S - Ps}{u^3\left(h^2 + 2\int\dfrac{T}{u^3}dv\right)} = 0,$$

$$\left[\text{since } \frac{s}{u}du + u.d\left(\frac{s}{u}\right) = d\left(\frac{s}{u}.u\right) = ds\right].$$

If we now recapitulate, after these transformations, the resulting differential equations, they will be

$$dt = \frac{dv}{u^2\sqrt{\left(h^2 + 2\int T\frac{dv}{u^3}\right)}} \ldots\ldots\ldots\ldots\ldots [a],$$

$$^{*}\,\frac{d^2u}{dv^2} + u + \frac{T\dfrac{du}{dv} - Pu}{u^3\left(h^2 + 2\int T\dfrac{dv}{u^3}\right)} = 0 \ldots\ldots\ldots [b],$$

$$\frac{d^2s}{dv^2} + s + \frac{T\dfrac{ds}{dv} + S - Ps}{u^3\left(h^2 + 2\int T\dfrac{dv}{u^3}\right)} = 0 \ldots\ldots\ldots [c],$$

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* These equations are the same, in fact, as what Laplace has given in *Mec. Cel.* 2$^{\text{de}}$ *Partie.* Liv. VII. p. 181: Thus, [*b*] under a different form is

$$\left(\frac{d^2u}{dv^2} + u\right) \cdot \left(1 + \frac{2}{h^2}\int T\frac{dv}{u^3}\right) + \frac{T}{h^2u^3} \cdot \frac{du}{dv} - \frac{P}{h^2u^2} = 0,$$

or, substituting for T and P their values, (see p. 67,)

$$\left(\frac{d^2u}{dv^2} + u\right) \cdot \left(1 + \frac{2}{h^2}\int \frac{d\mathcal{Q}}{dv} \cdot \frac{dv}{u^2}\right)$$

$$+ \frac{du}{h^2u^2 \,.\, dv} \cdot \frac{d\mathcal{Q}}{dv} - \frac{1}{h^2} \cdot \frac{d\mathcal{Q}}{du} - \frac{s}{h^2u} \cdot \frac{d\mathcal{Q}}{ds} = 0,$$

the third equation (*c*) also becomes

$$\left(\frac{d^2s}{dv^2} + s\right)\left(1 + \frac{2}{h^2}\int T\frac{dv}{u^3}\right) + \frac{1}{h^2u^3}T\frac{ds}{dv} + \frac{S}{h^2u^3} - \frac{Ps}{hu^2} = 0,$$

or neglecting $\left(\frac{d^2s}{dv^2} + s\right)\frac{2}{h^2}T\frac{dv}{u^3}$ which must involve the square of the disturbing force, since $\frac{d^2s}{dv^2} + s = 0$, when there is no disturbing force,

$$\frac{d^2s}{dv^2} + s + \frac{1}{h^2u^2} \cdot \frac{d\mathcal{Q}}{dv} \cdot \frac{ds}{dv} - \frac{1 + s^2}{h^2u^2} \cdot \frac{dQ}{ds} - \frac{s}{h^2u} \cdot \frac{d\mathcal{Q}}{du} = 0,$$

$$\text{or } \frac{d^2s}{dv^2} + s - \frac{r^2}{h^2(1+s^2)} \cdot \frac{ds}{dv} \cdot \frac{dR}{dv} + \frac{r^2}{h^2} \cdot \frac{dR}{ds} + \frac{sr}{h^2\sqrt{(1+s^2)}} \cdot \frac{dR}{dv} = 0.$$

which, when there is no disturbing force, are reduced to

$$dt = \frac{dv}{hu^2} \dots\dots\dots\dots\dots\dots\dots\dots [\alpha],$$

$$\frac{d^2u}{dv^2} + u - \frac{\mu}{h^2(1+s^2)^{\frac{3}{2}}} = 0 \dots\dots\dots [\beta],$$

$$\frac{d^2s}{dv^2} + s = 0 \dots\dots\dots\dots\dots\dots\dots\dots [\gamma],$$

since $T = 0$, $P = \mu u^2 \cos.^3 \phi = \dfrac{\mu u^2}{(1+s^2)^{\frac{3}{2}}}$, and

$$S = \mu u^2 \cos.^3 \phi \tan. \phi = \frac{\mu u^2 s}{(1+s^2)^{\frac{3}{2}}}, \text{ whence}$$

$$S - Ps = 0.$$

We have thus, under one point of view, two sets of equations: the latter belonging to the problem of two bodies, admitting of a complete integration, and establishing, by the results of that integration, the three Laws of Kepler: whilst the former, belonging to the problem of the three bodies, are capable, under certain restrictions, of only an approximate integration, and exhibit, by their results, certain derangements of Kepler's Laws, dependent, as to their cause, on the external and disturbing action of a third body.

The equations [*b*] and [*c*] are, it is plain, under a similar form. From the integration of one we could pass to that of the other: so that, the solution of the problem of the three bodies depends, in fact, on the integration of an equation such as

$$\frac{d^2u}{dv^2} + u + \Pi = 0.$$

This equation has been already (see p. 23.) integrated in two cases; when Π was either nothing, or was represented by a constant quantity: that is, when the equation belonged to the system of two bodies only: and, on this first integration, as a basis, we shall found a second, when Π is a function of u and v, and when the equation belongs to the system of three bodies.

The method by which the solution in the simple case is extended to the solution of the complicated, is technically called The *Variation of the Parameters.*

It is a method, as Lagrange observes, almost of equal importance with the Differential Calculus; and it seems to be singularly adapted for deducing the motions of a body revolving round one body and disturbed by a third, from those of a body simply revolving round another. For this end, instead of supposing, as in the simple case, (see p. 23.) the arbitrary quantities to be constant, it is merely necessary to consider them as variable, and to determine their variation by the *analytical difference* in the statement of the two cases. This is the principle of the method and solution, and which we will now exemplify.

Let $\frac{d^2 u}{d v^2} + u = 0$;

then, the solution is (see p. 23.),

$u = a \sin. v + b \cos. v$, ($a$, b being constant),

Now, of the equation $\frac{d^2 u}{d v^2} + u + \Pi = 0$;

let $u = a \sin. v + b \cos. v$,

be supposed to be the solution, a and b (technically called the *parameters*) being now variable, then

$$\frac{du}{dv} = a \cos. v - b \sin. v + \frac{da}{dv} \sin. v + \frac{db}{dv} \cos. v.$$

Make $\frac{da}{dv} \sin. v + \frac{db}{dv} \cos. v = 0$ .... [m],

and take the differential of $\frac{du}{dv}$, then

$$\frac{d^2 u}{d v^2} = -(a \sin. v + b \cos. v) + \frac{da}{dv} \cos. v - \frac{db}{dv} \sin. v.$$

Hence, $\frac{d^2 u}{d v^2} + u + \Pi = + \frac{da}{dv} \cos. v - \frac{db}{dv} \sin. v + \Pi = 0.$

By means of this last, and of the equation [m], determine da and db, according to the common method of elimination; then

$$da = - \Pi \cos. v . dv,$$
$$db = + \Pi \sin. v . dv,$$

and, integrating,

$$a = c - \int \Pi \cos. v . dv,$$
$$b = c' + \int \Pi \sin. v . dv.$$

Now, when $\Pi = 0$, the values of a and b are those constant ones which belong to them in the first and simple case: let a and b still denote them, then, $c = a$, and $c' = b$; therefore, if a' and b' denote their values when variable, we have

$$a' = a - \int \Pi \cos. v\, d\, v,$$
$$b' = b + \int \Pi \sin. v\, d\, v,$$

and accordingly,

$$\begin{aligned} u &= a' \sin. v + b' \cos. v \\ &= a \sin. v + b \cos. v \\ &\qquad - \sin. v \int \Pi \cos. v\, d\, v + \cos. v \int \Pi \sin. v\, dv, \end{aligned}$$

which is the form of the solution of the differential equation to which it was originally reduced by Clairaut (see *Mem. Acad.* 1748. pp. 421, &c. and *Theorie de La Lune*, ed. 2. p. 6.), and under which Laplace, the latest of Writers on Physical Astronomy, has exhibited it, (see *Mec. Celeste.* Liv. II. pp. 240, &c. also Cousin, *Astron. Phys.* pp. 23, 233. *Mem. Turin.* Tom. III. pp. 262, &c. Simpson's *Tracts*, pp. 92, &c. Dalembert, *Theorie de la Lune*, p. 25.)

If Π be a constant quantity, $-\frac{\mu}{h^2}$, for instance, then

$$-\sin. v \int \Pi \cos. v . dv + \cos. \int \Pi \sin. v . dv = \frac{\mu}{h^2} \sin.^2 v + \frac{\mu}{h^2} \cos.^2 v = \frac{\mu}{h^2};$$

therefore, of the equation

$$\frac{d^2 u}{d v^2} + u - \frac{\mu}{h^2} = 0,$$

$$u = a \sin. v + b \cos. v + \frac{\mu}{h^2},$$

is the solution, as before in p. 23.

Hence, if $\Pi = -\frac{\mu}{h^2} - \Omega$, Ω expresses what is due solely to the disturbing force, and,

$$\begin{aligned} u &= a \sin. v + b \cos v + \frac{\mu}{h^2} \\ &\qquad - \sin. v \int \Omega . \cos. v . dv + \cos. v \int \Omega \sin. v . dv, \end{aligned}$$

$$\text{or (see p. 25.)} = \frac{1}{a.(1-e^2)}(1 + e.\cos. v)^*,$$

$$- \sin. v \int \Omega \cos. v\,dv + \cos. v \int \Omega \sin. v\,.\,dv.$$

We easily then obtain the solution of the equation when Π is represented by a constant quantity; which is the case when there is no disturbing force.

Now the case next, in point of simplicity, to this, is that in which Π should be expounded by any cosine or sine of a multiple of the arc v. For instance, if Π should equal $A \cos. mv$, then, since the integrals of $A \cos. mv . \cos. v . dv$, and of $A \cos. mv . \sin. vdv$ can immediately be found, we are able at once to divest the value of u of its integral sign, and to exhibit it under a definite form. On this ground and consideration, that is, on the perception of the possibility of integrating the equation, if Π were represented by a term such as $A \cos. mv$, or by a series of terms such as $A \cos. mv + B \cos. nv +$ &c. the first mathematicians who treated of Physical Astronomy by the analytical method, turned their attention to the expounding and representing of Π by such a term, or by a series of such terms. For, that difficulty mastered, they would of course become possessed of the solution of the *Problem of the three bodies*. And to that point in the enquiry we shall soon advance; previously, however, it is necessary to exhibit the value of u when Π is represented by a term such as $A \cos. mv$.

Substitute, instead of $\cos. mv . \sin. v$,

$$\frac{1}{2}\,[\sin. (m+1)v - \sin. (m-1)v],$$

and, instead of $\cos. mv . \cos. v$,

$$\frac{1}{2}\,[\cos. (m-1)\,v + \cos. (m+1)\,v],$$

(see *Trig.* ed. 2. p. 26.), then integrate (*Trig.* p. 98.), and on reducing the expression, there will result

$$A . \cos. v \int \cos. mv \sin. v . dv - A \sin. v \int \cos. mv . \cos. v . dv =$$

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* We must be careful not to confound a the semi-axis major with the *arbitrary quantity* a.

$$\frac{A\cos. mv}{m^2-1}-\frac{A\,.\cos. v}{m^2-1},$$

(see *Trig.* ed. 2. pp. 104, &c.) *.

Hence,

$$u=a\sin. v+b\cos. v+\frac{A\,.\cos. mv}{m^2-1}-\frac{A\,.\cos. v}{m^2-1}.$$

Let $\Pi = A\cos. mv + B\cos. pv + C\,.\cos. qv$, then there will result

$$u=a\sin. v+b\,.\cos. v+\frac{A\,.\cos. mv}{m^2-1}+\frac{B\cos. pv}{p^2-1}$$

$$+\frac{C\,.\cos. qv}{q^2-1}-\left(\frac{A}{m^2-1}+\frac{B}{p^2-1}+\frac{C}{q^2-1}\right)\cos. v.$$

If B, C, should $=0$, and if $A=\frac{\mu}{h^2}$, and $m=0$, then

$$u=a\sin. v+b\cos. v+\frac{\mu}{h^2}-\frac{\mu}{h^2}\cos. v$$

$$=a\sin. v+\left(b-\frac{\mu}{h^2}\right)\cos. v+\frac{\mu}{h^2},$$

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* The use of Treatises on pure Science is to demonstrate methods, and to prepare them for their application to *Physicks*. It is sufficient, in general, to refer to such treatises when use is made of any of the methods which they contain. If we were obliged to demonstrate every theorem or formula which Physical Astronomy requires the aid of, the bulk of a treatise on that science would be enormous. And, in the present Treatise, mere references would more frequently have been substituted instead of demonstrations, could they have been made to Works in the English language. But, unfortunately, what are reckoned our most profound treatises on pure mathematics, cannot be brought to bear on Physical Science; their theorems are so abstruse as to be altogether withdrawn from purposes of utility; and the expenditure of time and thought, which they must have cost, can be viewed with complacency only by conjecturing (by the doubtful light of imperfect analogies) that such recondite processes, now worth nothing, may, at some future state of science, possess real value.

which is exactly of the *same form* (since $b - \frac{\mu}{h^2}$ is, instead of b, the indeterminate coefficient) as the result in p. 98. and in specific instances, would assign to the coefficients of sin. v and cos. v the same numerical values.

Hence, if of

$$\frac{d^2 u}{d v^2} + u - \frac{\mu}{h^2} = 0,$$

(Π being $= 0$), $u = \frac{1}{a(1-e^2)}(1 + e \cos. v)$ (see p. 25.)

is the integral, then of

$$\frac{d^2 u}{d v^2} + u - \frac{\mu}{h^2} + A \cos. m v + B \cos. p v + \&c.*$$

$$u = \frac{1}{a(1-e^2)}(1 + e \,.\, \cos. v) + \frac{A}{m^2 - 1} \cos. m v + \frac{B}{p^2 - 1} \cos. p v + \&c.$$

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* We have solved the above equation by the direct method of the *Variation of Parameters;* which method will again be resorted to when the *variability* of the elements of a planet's orbit is treated of. But it would have been easy to have arrived at the above solution in the following manner:

Let $u = a \cos. N v - K + A \cos. m v + B \cos. p v$;

then $\frac{d^2 u}{d v^2} = - a N^2 \cos. N v - A m^2 \cos. m v - B p^2 \cos. p v,$

$$\frac{d^2 u}{d v^2} + N^2 u = - K N^2 + A (N^2 - m^2) \cos. m v + B (N^2 - p^2) \cos. p v,$$

or, of $\frac{d^2 u}{d v^2} + N^2 u + K N^2 + A (m^2 - N^2) \cos. m v + B (p^2 - N^2) \cos. p v = 0,$

$$u = a \cos. N v - K + A \cos. m v + B \cos. p v,$$

is the solution; reversely, therefore,

$$u = a \,.\, \cos. N v - \frac{K}{N^2} + \frac{A}{m^2 - N^2} \cos. m v + \frac{B}{p^2 - N^2} \cos. p v,$$

is the integral equation of

$$\frac{d^2 u}{d v^2} + N^2 u + K + A \cos. m v + B \cos. p v,$$

which agrees, essentially, with the solution in the text on putting $N = 1$.

$$-\left(\frac{A}{m^2-1}+\frac{B}{p^2-1}+\&c.\right)\cos v$$

is the integral. The former is the value of the inverse of the radius vector in the system of two bodies, and the latter would be its value in that of three, if the disturbing force $\left(\pi-\frac{\mu}{h^2}\right)$ could be represented by a series of terms such as

$$A\cos.\,mv+B\cos.\,pv+\&c.$$

If we put the preceding equation under this form

$$u=\frac{1}{a(1-e^2)}+\left(\frac{e}{a(1-e^2)}-\frac{A}{m^2-1}-\frac{B}{p^2-1}\right)\cos.\,v,$$
$$+\frac{A\cos.\,mv}{m^2-1}+\frac{B\cos.\,pv}{p^2-1};$$

then, were it not for the two last terms $\frac{A\cos.\,mv}{m^2-1}$, and $\frac{B.\cos.\,pv}{p^2-1}$, the equation would be that of an ellipse in which the eccentricity, from its value ae, would be changed into $a'e-\frac{A.(1-e^2)a'}{m^2-1}-\frac{B(1-e^2)a'}{p^2-1}$, a' being the semi-axis of of the new ellipse. The effect then of the disturbing force (on the supposition that it can be represented by $A\cos.\,mv+B\cos.\,pv$) is to destroy the elliptical form of the orbit, and to cause that curve to be described, which is indeed of no denomination, nor of any known property, and is solely designated and characterised by the analytical equation

$$u=\frac{1}{a(1-e^2)}(1+e\cos.\,v)+A\cos.\,mv+\&c.$$

This curve, however, if the disturbing force be small (which it is in every real application of the problem of the three bodies) differs but little from an ellipse; not much from that ellipse which is described when no disturbing force acts, and of which the equation is

$$u=\frac{1}{a.(1-e^2)}(1+e\cos.\,v);$$

but less from that other ellipse (see p. 102. l. 8.) of which the equation is

$$u = \frac{1}{a\,(1-e^2)} \left[1 + \left(e - \frac{A\,a\,(1-e^2)}{m^2-1} - \frac{B\,a\,(1-e^2)}{p^2-1}\right) \cos.\, v\right]$$

This second equation, as we shall hereafter see, will serve to explain, to a considerable degree, Newton's object in the ninth Section of his *Principia*.

The present Chapter, since its object (which is the integration of a differential equation) is attained, might here terminate; but we wish, previously to closing it, to direct, for a short time, the student's attention to the *Elliptical Theory* geometrically treated, and to a sort of analogy that exists, or that may be fancied to exist, between some points of that theory and of the analytical method which has been just described.

In the geometrical method then of treating the subject, the ellipse is considered as the standard and genuine curve, from which the real curve that is described, differs in consequence of the disturbing force, and slightly differs by reason of the minuteness of that force. It is the plan therefore in that method to set out from the ellipse, and to investigate the aberration from it. In the analytical method, the equation first established, and from which we enter on deeper researches, is that belonging (in technical phraseology) to the *system of two bodies*. The form of the resulting value of u (the inverse of the radius vector) the exact value in

$$\frac{d^2 u}{d v^2} + u - \frac{\mu}{h^2} = 0$$

is assumed, in order to deduce an approximate one belonging to

$$\frac{d^2 u}{d v^2} + u + \Pi = 0,$$

which is the real equation that requires solution, and belongs to the *system of three bodies*.

In both the methods (as they are practised) the smallness of the disturbing force is an essential condition. In the geometrical, the body's path is almost in an ellipse, which it could not be, were the disturbing force large. In the analytical, the process of approximating to the value of u depends on the minuteness of $\Pi - \frac{\mu}{h^2}$.

The object of this Chapter has been said to be the integration of the equation

$$\frac{d^2 u}{d v^2} + u + \Pi = 0,$$

which cannot generally be accomplished; but we have arrived at this important result, namely, the practicability of integrating it, if Π could be represented by a series of the sines or cosines of multiple arcs. By that route then * we have a chance of arriving at our object; we have got something of a clue, and our next steps, it should seem, ought to be directed to the conversion of Π into a series of such sines or cosines.

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* This, however, is not the sole route by which the integration of the equation is to be arrived at. It would be attained if Π could be represented by $K + n u$, and in some cases, it may nearly be represented by such a quantity. For instance, if Π should equal $L + M u^m + N u^n$, then, the orbit being nearly circular, and, consequently, u nearly $= \frac{1}{a}$ (a being the mean distance,) we should have

$$u^m = \left[\frac{1}{a} - \left(\frac{1}{a} - u\right)\right]^m = \frac{1}{a^m} - \frac{m}{a^{m-1}} \times \left(\frac{1}{a} - u\right), \text{ nearly,}$$

by neglecting the higher powers of $\frac{1}{a} - u$; and the approximate value for u^n would be of a similar form. Hence, Π would be of the form $K + n u$, and consequently the differential equation would be

$$\frac{d^2 u}{d v^2} + (1 + n) u + K = 0,$$

the integration of which by the note of p. 101. is

$$u = a \cos. \surd(1 + n) u - \frac{K}{1 + n}.$$

This method, however, is a partial one; that is, it obtains only in particular circumstances, and, besides, its results are included amongst those of the general one, and in which Π is represented by a series of cosines.

It is necessary, however, to examine certain circumstances that are adjacent to this part of the main route of investigation, before we proceed along it. These circumstances are peculiarities of solution, which, in certain predicaments, attach themselves to that method of approximation which has been described in the present Chapter. They are (and under this point of view we shall first consider them) analytical. But, in the application of the Calculus to the subject of this Treatise, they produce certain incongruities which vitiate that explanation of the Planetary Theory, which is founded on the principles of Physical Astronomy. It is necessary, therefore, to get rid of them: to shew why they vitiate, and how, by a modification of the Calculus, they may be made not to vitiate that explanation. The student, however, who, at this point of the enquiry, shall feel no inclination to attend to these peculiarities, may, in his first perusal, pass over the succeeding Chapter.

CHAP. VIII.

On certain Ambiguities of Analytical Expression that occur in the Problem of the Three Bodies; their Source and Remedy. A new Form for the Integral value of u *from which the Arcs of Circles are excluded. Consideration on the Alteration which certain small Quantities may receive from the Process of Integration. Comparison between the Analytical Formulæ, and the Results of the Geometrical Method. Observations on the Ninth Section of the Principia.*

IT has already appeared (see p. 99.), that if, in the expression for the disturbing force, a term such as $A \,.\, \cos.\, m v$ should enter, the methods of integration would introduce into the expression of the value of u, this term

$$\frac{A \,.\, (\cos.\, m v - \cos.\, v)}{m^2 - 1}.$$

If we make $m = 1$, then, both the numerator and denominator are $= 0$, and the term becomes

$$\frac{A \times 0}{0},$$

which is an useless result. This is said to be a fault of calculation (*faute du Calcul*); but, if we examine the matter, it will appear that the above indefinite expression arises entirely from an *extension** of a rule. Thus (see p. 100.), the result of the integration gives

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* There are many similar instances to be found in Analytical Science. The integral of $\int \frac{d x}{x}$ is a case in point. The rule for finding its integration cannot extend to that case, because in the enumeration of cases where the rule holds and is good, that particular one cannot be comprehended.

$$\frac{A\,.\,(\cos.\,m\,v - A\,.\,\cos.\,v)}{m^2 - 1},$$

where m is supposed to represent any number. But the value $m = 1$ must be excluded, and precisely for this reason; that, if we suppose the expression for the disturbing force to contain a term such as $A\,.\,\cos.\,v$, the corresponding terms introduced into the value of u, by the process of integration, do not assume that form which belongs to them when m is expounded by any number 2, 3, 4, &c. and which therefore is restricted in its generality by the exception of the case in which $m = 1$.

In order to find the result of the integration in this particular case of $m = 1$, we must substitute $\cos.\,v$ instead of $\cos.\,m\,v$ (see p. 98.); in which case, the quantity to be integrated will be

$$\cos.\,v\int\cos.\,v\,.\,\sin\,v\,.\,d\,v - \sin.\,v\int\cos.^2\,v\,.\,d\,v =$$

(see *Trig.* ed. 2. pp. 26. 36.)

$$-\cos.\,v\,.\,\frac{\cos.\,2\,v}{4} - \sin.\,v\,.\left(\frac{v}{2} + \frac{\sin.\,2\,v}{4}\right) =$$

$$-\frac{\cos.\,v}{4} - \frac{v}{2}\,\sin.\,v.$$

Hence, the integral of

$$\frac{d^2 u}{d\,v^2} + u + A\,.\,\cos.\,v = 0, \text{ is}$$

$$u = a\,\sin.\,v + b\,.\,\cos.\,v - \frac{A\,.\,v}{2}\,\sin.\,v.$$

Here then, although not by deduction from the general formula, we have the expression for u when Π is expounded by $A\,\cos.\,v$, and the Calculus, if it can be said to have been faulty, is completely amended. We have, however, now to consider, whether any incongruity will be attached to the above peculiar value of u in its application to the Lunar Theory, or to the problem of the three bodies.

Now, between the general value of u (see p. 100.) and the preceding peculiar one, there is this notable difference; that, in the latter, the arc v appears *without the sign of the sine*; the consequence of which is, that $\frac{A\,v}{2}\,\sin.\,v$ will, on the whole, by the increase of v, continually increase; and the value of u will at the end of any period be different from that which it was at the beginning, and more different, the greater the period. Preceding values of the radius vector $\left(=\frac{1}{u}\right)$ therefore cannot recur, and the curve traced out by the extremity of the radius vector cannot be of an oval form or *re-entering**. But it is clear from observation that the orbits of the planets are oval: their radii vectores, therefore, cannot be generally expounded by an expression such as the inverse of

$$a \, . \sin. v + b \, . \cos. v - \frac{A}{2} v \sin. v,$$

and consequently the Calculus is here also in fault, or the Planetary phenomena are not explicable on the preceding premises.

If we examine the process of p. 99. it will immediately appear that, should such a term as $A \cos. v$ enter into the composition of Π, the process of integration must introduce the term $-\frac{A}{2} v \, . \sin. v$. This point being certain and determined, we are naturally led to enquire whether in expounding the disturbing force we must of necessity use such a term as $A \cos. v$, or whether such a term finds its way into the expression for Π by that peculiar method of approximate integration, to which, from want of ampler resources, the state of analytical science obliges us to resort.

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* 'On voit par la que lorsque Ω, renfermera des termes de cette espece, l'equation de l'orbite contiendra des angles v; et quelques petits que soient les termes ou ils entrent, ils peuvent donner les plus fortes corrections a la valeur de r, lorsqu'on supposera l'angle v d'un grand nombre de revolutions. Ainsi si l'on n'a rien neglige en determinant Ω, on sera sûr l'orbite s'ecartera a la fin fort considerablement d'une ellipse et changera entierement de forme'. Clairaut, *Theorie de la Lune*, ed. 2. p. 11. See also Dalembert. *Theorie de la Lune*, pp. 30, 34, &c.

The general equation (see p. 96.) on substituting $-\frac{\mu}{h^2} - \Omega$, instead of Π, is

$$\frac{d^2 u}{d v^2} + u - \frac{\mu}{h^2} - \Omega,$$

in which Ω is due entirely to the disturbing force; and, if (see p. 95.) we examine the composition of Ω, it will appear to be a function of u and v. Ω, therefore, cannot be expressed in terms of v except u can be expressed in terms of the same quantity. But u, in the problem of the three bodies, to which the preceding differential equation belongs, is to be determined by the integration of that equation; which cannot be accomplished except Ω be determined. Here then we are entangled in what is called, a *Vicious Circle;* but, by attending to the mode by which we are obliged, in this as in other cases, to escape from it, we shall perceive the reason why a term such as A cos. v accompanies the result.

Now the mode is to assume an approximate value of u, to substitute such value in Ω, and then to obtain a more correct value of u by the solution of

$$\frac{d^2 u}{d v^2} + u - \frac{\mu}{h^2} - \Omega = 0.$$

Let us take a case (for the principle is the same in all instances): suppose T the tangential force to be nothing, and P, the force in the direction of the radius (see pp. 10, 95.) to equal $\mu u^2 + \frac{K}{u}$;

$$\text{then } \Pi = -\frac{P}{h^2 u^2} = -\frac{\mu}{h^2} - \frac{K}{h^2 u^3};$$

$$\text{and consequently, } \Omega = \frac{K}{h^2 u^3}.$$

Now, if the first value which we assume for u be its elliptical value, or (see p. 23.) that which results from the integration of

$$\frac{d^2 u}{d v^2} + u - \frac{\mu}{h^2} = 0,$$

we shall have, making $p = a.(1 - e^2)$,

$$u = \frac{1}{p}(1 + e.\cos. v).$$

Substitute this in the value of Ω, and

$$\Omega = \frac{Kp^3}{h^2.(1 + e.\cos. v)^3} = \frac{Kp^3}{h^2}(1 - 3e.\cos. v),$$

if e be supposed small, and we neglect the terms that involve e^2, e^3, &c. The differential equation now then becomes

$$\frac{d^2 u}{d v^2} + u - \frac{\mu}{h^2} - \frac{Kp^3}{h^2} + \frac{3Kp^3 e}{h^2}\cos. v = 0,$$

in which the last term corresponds to $A \cos. v$. This equation being integrated by the method described in p. 98, &c. the value of u must contain this term.

$$-\frac{3Kp^3}{2h^2}e.\frac{v}{2}.\sin. v,$$

which, as we have already remarked, vitiates the result.

But the result is vitiated, or v *without* the signs (*hors des signes*) of sine and cosine, is introduced solely by that peculiar method of approximation which has been adopted: for, instead of assuming for u its elliptical value, let us make

$$u = \frac{1}{p} - \left(\frac{1}{p} - u\right);$$

then, neglecting terms such as $\left(\frac{1}{p} - u\right)^2$, $\left(\frac{1}{p} - u\right)^3$, &c. which, in orbits nearly circular, are very small,

$$u^{-3} = p^3(4 - 3pu),$$

and consequently, the equation of p. 109. l. 19. becomes

$$\frac{d^2 u}{d v^2} + u - \frac{\mu}{h^2} - \frac{4Kp^3}{h^2} + \frac{3Kp^4}{h^2}u = 0;$$

$$\text{or, } \frac{d^2 u}{d v^2} + \left(1 + \frac{3Kp^4}{h^2}\right)u - \frac{\mu}{h^2} - \frac{4Kp^3}{h^2} = 0.$$

Now the integral of this equation, by the note of p. 101., is

$$u = a \,.\, \cos. \sqrt{\left(1 + \frac{3Kp^4}{h^2}\right)} \,.\, v + \frac{1}{h^2}\,(\mu + 4Kp^3),$$

where a is an arbitrary quantity, the value of which is to be assigned on the principles laid down in pp. 23, &c.

The value of u may be put under this form

$$u = \frac{\mu + 4Kp^3}{h^2}\,(1 + e' \,.\, \cos.\, c\,v)$$

in which $e' = \dfrac{a\,h^2}{\mu + 4Kp^3}$,

and $c = \sqrt{\left(1 + \dfrac{3Kp^4}{h^2}\right)}$,

and thus its expression becomes similar to that elliptical value of u, namely,

$$u = \frac{\mu}{h^2}\,(1 + e \,.\, \cos.\, v),$$

and which, considered analytically, is the integral value of u in the differential equation

$$\frac{d^2 u}{d v^2} + u - \frac{\mu}{h^2} = 0.$$

Hence, the disturbing force $\left(\dfrac{K}{h^2 u^3}\right)$ changes the constant part $\dfrac{\mu}{h^2}$ to $\dfrac{\mu}{h^2} + \dfrac{4Kp^3}{h^2}$, the ratio of the eccentricity from e to e', and the angle v to $c\,v$, or $\sqrt{\left(1 + \dfrac{3Kp^4}{h^2}\right)}\,v$.

The solution, however, which has been obtained is an approximate one, depending on the smallness of the eccentricity of the orbit. In order to obtain greater exactness, we ought to pursue the method; that is, we must repeat the process and take account of some of those terms that were neglected in the first operation. Suppose that we retain $\left(\dfrac{3}{p} - u\right)^2$ and neglect the higher powers, then (see p. 110.),

$$u^{-3} = p^3\,(4 - 3pu) + 6p^5 \,.\, \left(\frac{1}{p} - u\right)^2,$$

and the differential equation becomes

$$\frac{d^2 u}{d v^2} + \left(1 + \frac{3 K p^4}{h^4}\right) u - \frac{\mu + 4 K p^3}{h^2} + \frac{6 p^5}{h^2}\left(\frac{1}{p} - u\right)^2 = 0.$$

Now if, in the last term, the value of u obtained by the preceding approximation be substituted, that is, if we assume

$$u = \frac{\mu + 4 K p^3}{h^2}\,(1 + e' \cos.\, c v),$$

and substitute it in $\frac{6 p^5}{h^2}\left(\frac{1}{p} - u\right)^2$, there will arise a term such as $A \cos.\, c v$: so that, in this second approximation, we should fall into that fault of expression which it was the object of the preceding process to avoid: for, if we revert to pp. 98, &c. it will immediately appear that the method of integration applied to

$$\frac{d^2 u}{d v^2} + c^2 u - \frac{\mu + 4 K p^3}{h^2} + \&c. + A \cos.\, c v,$$

must produce a term such as

$$B \,.\, \frac{v}{2} \,.\, \sin.\, c v.$$

This relapse, however, is to be obviated, by means similar to what have already (see pp. 110.) been employed. Instead of substituting in the terms expressing the disturbing force the last obtained value of u, expand those terms, and add the term involving u to the term which already involves it in the differential equation: thus, suppose the term involving u and resulting from $\frac{6 p^5}{h^2}\left(\frac{1}{p} - u\right)$ to be $N^2 u$, then the preceding equation (l. 2.) may be thus written

$$\frac{d^2 u}{d v^2} + (c^2 + N^2)\, u - \frac{\mu + 4 K p^3}{h^2} + \&c. = 0.$$

and the integral of this (see pp. 100, &c.) will be of the form

$$u = \alpha \cos.\, \surd(c^2 + N^2) \,.\, u + L,$$

and, by similar artifices, we may continue the approximation to the value of u and still exclude terms containing *arcs without the sign* from the result [*].

We may now consider this point of the Calculus as settled. We have shewn first, (see p. 106, &c.) why the analytical expression becomes apparently insignificant: next, (see p. 108.) how the removal of that fault induces, when we look to the practical application of the result, another of equal or greater importance: in the third place, (see p. 110.) we have made manifest the source of this last error: and lastly, (see pp. 110, &c.) we have explained the method of avoiding it. The Calculus, therefore, though liable to ambiguity, is relieved from the effect of it, and, as far as we have advanced, shewn not to be incompetent to the explanation of the Planetary Phenomena on the Principles of Physical Astronomy.

But the preceding error, which we have been considering, is not the only one that originates from the method of approximation. We will now examine another, not of ambiguity, but of a different kind, which the imperfection of that method renders us liable to.

In the most simple and ordinary processes of approximation,

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[*] The method here prescribed corresponds to, and is the same, in effect, as the rule given by Dalembert, *Theorie de la Lune,* pp. 36, 37. For he says, if

$$u = H + L\cos. cv,$$

and if the term producing the arc in Π be $\gamma . \cos. cv$; then, in order to exclude the arc from the result, we must state the equation,

$$\frac{d^2u}{dv^2} + c^2u + \Pi = 0,$$

under this form,

$$\frac{d^2u}{dv^2} + \left(c^2 + \frac{\gamma}{L}\right)u + \left(\Pi - \frac{\gamma}{L}u\right) = 0,$$

which is a contrivance the same as that which is explained in the preceding pages of the text. See also on this subject. Laplace, *Mem. Acad.* 1772. Part II. p. 267. Cousin, p. 235: which latter author quotes from Lagrange's Memoir in the *Miscellanea Taurinensia,* tom. III. p. 263.

which are not connected with processes of integration, we may, in the first instance, safely reject quantities according to their degrees of smallness. But the case is quite different, when, in some stage or other of the calculation, certain quantities that enter therein are to be integrated. For, by integration quantities acquire divisors; and if the divisors be very small, the quantities, although minute previously to integration, may not be so after it. If (in order to exemplify this point), we suppose

$$P = A + \alpha . \cos. m x + \beta \cos. n x + \gamma \cos. r x + \&c.$$

then, if the coefficient γ should be much smaller than A, α and β, the approximate value of P, would be

$$P = A + \alpha . \cos. m x + \beta . \cos. n x.$$

But if, instead of the first equation, we had this

$$P = \int d x . (A + \alpha . \cos. m x + \beta \cos. n x + \gamma \cos. r x + \&c.),$$

then we cannot safely reject the term $\gamma \cos. r x$ except we first ascertain the magnitude of r: for if r should be small, then, after integration, the term

$$\int d x . \gamma \cos. r x = \frac{\gamma}{r} \sin. r x,$$

might be either not less or much greater (according to the value of r) than $A x$, $\frac{\alpha \sin. m x}{m}$, or $\frac{\beta \sin. n x}{n}$: in which case the rejection of the small term $\gamma \cos. r x$ in the first instance, and the consequent assumption of the equation,

$$P = \int d x \, (A + \alpha . \cos. m x + \beta \cos. n x),$$

would lead to erroneous results.

This is an analytical illustration of the circumstance that must be attended to. But, instead of the former instance, we might have taken the one in p. 100, and the inference to be drawn from it would have been of the same nature. Thus, let

$$\Omega = M . \cos. m v + N . \cos. n v + Q . \cos. q v,$$

then the equation,

$$\frac{d^2 u}{d v^2} + u + \Omega = 0,$$

integrated by the method described in pages 100, &c. contains, amongst other, these quantities,

$$\frac{M \cdot \cos. m v}{m^2 - 1}, \quad \frac{N \cdot \cos. n v}{n^2 - 1}, \quad \frac{Q \cdot \cos. q v}{q^2 - 1}.$$

Hence, as it has been shewn before, although, Q being a smaller coefficient than M and N,

$$M \cdot \cos. m v + N \cdot \cos. n v,$$

should be very nearly the value of Ω, yet the assumption of that approximate value might be productive of considerable and important error in determining the value of u: for that value, by virtue of the double integration, contains a term $\frac{Q}{q^2 - 1} \cos. q v$* which, if $q = 1$ nearly, may be of considerable magnitude, and, it is easy to see, of such magnitude as to be greater than either $\frac{M}{m^2 - 1} \cos. m v$, or $\frac{N}{n^2 - 1} \cos. n v$; so that the first and apparently safe rejection of the relatively small quantity $Q \cos. q v$, may prove to be the virtual detention of the smaller terms to the exclusion of the larger.

Several curious points in the Planetary Theories depend on the above principle. It enabled M. Laplace to explain (what had long embarrassed Astronomers) the *retardation* of Saturn's mean motion. In the theory of that planet disturbed by Jupiter there occur terms depending on the *argument* (see *Astronomy*, p. 324.) $5 n't - 2nt + A$, t being the time, n', and n the mean motions of Saturn and Jupiter; now, such terms in the differential equation involve the cubes of the eccentricities of the orbits, and consequently, (since $e^3 = .0001118$, and $e'^3 = .0001772$), are exceedingly small: but by integration they assume divisors such as $5n' - 2n$, $(5 n' - 2n)^2$ which also are exceedingly small, since five times Saturn's mean motion ($5 n'$) is nearly equal twice Jupiter's ($2 n$).

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* 'Il faudra donc avoir grande attention a toutes les termes de cette nature et y porter plus de scrupule que dans les autres, par rapport aux fractions qu'on negligera'. Clairaut, *Theorie de la Lune*, p. 12. See also Simpson's *Tracts*, p. 157.

The terms then after integration may become of sensible magnitude, or may expound inequalities large enough to be detected by observation.

It appears from the preceding cases, that it is not ambiguity of expression, but liability to actual error of computation that we must take caution against; and, the peril is to be met not by any artifice or contrivance drawn from the resources of calculation, but by a careful and minute examination of the terms to be rejected. The rejection of terms in the differential equation must always be made with reference to the form which they assume after integration.

These general maxims, however, are not carried into effect except with great difficulty. The process is not only one of successive approximation, but also of successive integration: it continually adds new terms to the result: it is therefore, not easy to see that terms proposed to be rejected in the beginning (and all the terms contained in Ω cannot be retained) shall not, towards the end, and by the effect of combination, become, or give rise to, terms of retainable magnitude.

The terms of the kind Q cos. qv, when q is nearly $= 1$ and that render the calculation liable to error, belong to the first equation

$$\frac{d^2 u}{d v^2} + u + \Pi = 0;$$

for, in the double integration of this equation, the above terms acquire very small denominators, such as $(q^2 - 1)$.

But in the second equation

$$dt = \frac{dv}{u^2 \sqrt{\left(h^2 + 2\int \frac{T\,dv}{u^3}\right)}},$$

terms of that kind need not to be attended to; for, by integration, they acquire denominators, such as q, and accordingly their values remain nearly the same when $q = 1$. But if q were very small, the case would be quite different; then, although minute before integration *, they might, by means of it, become large: and hence, it is easy to see, the terms of the second equation require the same careful examination as those of the first.

The denominator of the fraction expressing the value of dt (see p. 116.) contains this term $\int \frac{T\,dv}{u^3}$. If, therefore, $\frac{T\,dv}{u^3}$ should contain a term such as $Q \cos. q v$, and q should be a

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* That a term such as $Q \int \cos. q v \,.\, dv = \frac{Q}{q} \sin. q v$ cannot, in a process of approximation, be rejected, without an examination of its relative magnitude, is plain from merely analytical considerations. But Thomas Simpson in his Tracts, p. 157, treating of the Lunar Theory, has shewn, by a reference to physical causes, why it is necessary to retain such terms; and, after the following manner. A term such as $Q . \cos. q v$, may, as a correction, represent either an augmentation or a diminution: in either case it will continue to represent what it first represents, as long as $Q \cos. q v$ continues of the same sign. If q be small, $\cos. q v$ will continue of the same sign, till v becomes so large, that $q v = 90°$. Suppose, for instance, that $q = \frac{1}{8}$, then v must $= 720°$ before $\cos. q v$ can change its sign; or $Q \cos. q v$, if an augmentation, will continue to be such, till the body has described two entire revolutions. There will, therefore, during this time, be a continual accumulation of the effect which $Q . \cos. q v$ is meant to represent: and, although the momentary effect, expounded by $Q . \cos. q v$ may, from the smallness of Q, be very small, yet the accumulated effect, expounded by $Q \sin. q v$, may be considerable. The case, however, is different if q should be large, for then $\cos. q v$ would quickly change its sign and expound an effect of an opposite kind. If, for instance, q should $= 8$, then $Q \cos. q v$, if representing an augmentation, could only represent it, during the description of an angle $= 11° 15'$: after that, it would change its sign and represent a diminution, and again, after the description of $11° 15'$ an augmentation, and so on. In this case, therefore, instead of an *accumulation* of effects of the same kind, a *counteraction* of opposite effects takes place: and thence it must happen that the *general* or *mean* effect would remain the same, or would be but slightly altered, if that part of the disturbing force which is expounded by $Q \cos. q v$ were annulled.

small number, $\int \frac{T\,dv}{u^3}$ might contain a term of an order superior (with regard to magnitude) to $Q \cos. qv$.

Hence, it would seem to follow, since a double integration must be performed in order to obtain the time, that a quantity of the fifth order (supposing the quantities to be conventionally distributed into order) * might by its effect become of the third †. The fact however is, that when q is very small, the expression for the time does not contain any terms with denominators equal q^2, when only the first power of the disturbing force is taken account of, (see Laplace, *Mec. Celeste.* Part II. Liv. VII. pp. 190. 191.)

The chief object of the present Chapter is now attained. Its discussions are of a nature almost entirely analytical, but made on instances that really occur in the problem of the three bodies. We will, therefore, consider, whether any obvious inferences relating to Physical Astronomy can be drawn from them, or whether any connexion or analogy can be traced between the methods that have been adopted, and the peculiar methods of the founder of that science.

The simplest form of the differential equation is

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* Laplace in his *Mec. Cel. Partie* II. Liv. VII. p. 132. proposes to call the fraction $\frac{1}{13}$, expressive of the relation between the Sun's and Moon's mean motions, a very small quantity of the first order: and to class, under the same order, the eccentricities of the Solar and Lunar orbits, and their mutual inclination. Then, the squares and products of these are to be held as very small quantities of the second order: in this arrangement Laplace probably followed Dalembert, given in p. 43, &c. of his *Theorie de la Lune.*

† 'De toutes ces observations, il s'ensuit 1° que si dans la quantité $\frac{\pi\,dz}{u^3}\left(=\frac{T\,dv}{u^3}\right)$ il se rencontre des termes de la forme $\cos. kz$ ($= \cos. pv$) k etant une quantité forte petite de l'ordre de n, il faut pousser les coefficiens de ces termes jusqu'aux quantités infinitement petites de cinquieme ordre, puisque ces termes par la double integration seront abaissés jusqu' a n'etre plus qu'infinitement petites du troisiême ordre.' Dalembert, *Theorie de la Lune*, p. 47.

$$\frac{d^2 u}{d v^2} + u + \frac{\mu}{h^2} = 0,$$

which belongs to the problem of *two* bodies and the elliptical theory; and its integration gives us

$$u = \frac{1}{p} + \frac{e}{p} \cos. v,$$

the known equation, (see Vince's *Conic Sections*) of an ellipse.

If this value of u be substituted, as an approximate value of u, in the differential equation constructed by taking in some of the larger terms of the disturbing force, (supposed to vary as the distance and to act solely in the direction of the radius vector) there results, *when the method is corrected,* (see p. 111, &c.) an equation of this form

$$u = \frac{1}{L} + \frac{e'}{L} \cos. c v,$$

which, however, (see p. 110.), since some terms, by reason of the small eccentricity of the orbit, are neglected, is only an approximate value.

This last solution, considered as an analytical one, is similar to the former,

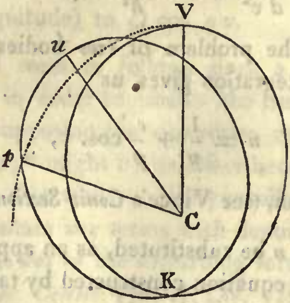
$$u = \frac{1}{p} + \frac{e}{p} \cos. v,$$

which is an equation to an ellipse, v being the angle contained between the axis major and the radius vector; but

$$u = \frac{1}{L} + \frac{e'}{L} \cos. c v,$$

is not an equation to an ellipse when v is the anomaly. Still, which is curious, the body's place, as determined by the preceding equation, can be found by means of a construction, of which an ellipse is the essential part. Thus,

Let V be the apside, C the centre of motion, and let the angle $CVp = v$: then, if $\frac{1}{Cp}$ $(= u)$ be assumed $= \frac{1}{L} + \frac{e'}{L} \cos. c v$, p is the body's place, and Vp described by p, (or the locus of the extremity of the line determined by the preceding equation) is part



of the body's orbit. Thus far is independent of an ellipse and of every other curve; we now come to the construction.

C being the focus, CV part of the axis major, describe an ellipse the semi-parameter of which shall $= L$, and the ratio of its eccentricity to the semi-axis shall $= e'$. Incline the line $Cu \, (= CV)$ so to CN, that the angle VCu shall equal $v - cv$, and on Cu describe an ellipse similar to that which has been already described on CV, then, since

$$uCp = VCp - VCu = v - (v - cv) = cv,$$

$$u \left(= \frac{1}{Cp}\right) = \frac{1}{L} + \frac{e'}{L} \cos. cv,$$

is the equation to the ellipse upK, and, accordingly, p the body's place is in such ellipse.

By means of this device then and construction, the body may be supposed to be always moving in a *moveable* ellipse. And it was under this point of view that Newton, when he made the first modification of, or departure from, the strictly *elliptical* theory, considered the planetary motions, (see *Princ.* Sect. IX.).

The preceding construction does not obtain, except (see pp. 110, &c.) such terms as $\left(\frac{1}{p} - u\right)^2$, &c. are rejected: it is only true, therefore, in orbits of very small eccentricity *. There is one variation of the disturbing force, however, in which the construction will hold, whatever be the eccentricity of that orbit: if that force varies inversely as the cube of the distance, or, if $P = \mu u^2 + K u^3$, then (see pp. 109, &c.) $\Omega = \frac{K u}{h^2}$, and the differential equation is exactly

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* Circulis maximi finitimi. *Princ.* Sect. IX. Prop. XIV.

$$\frac{d^2 u}{d v^2} + \left(1 - \frac{K}{h^2}\right) u - \frac{\mu}{h^2} = 0,$$

the integral of which is

$$u = \frac{\mu}{h^2}\left[1 + e \, . \cos. \sqrt{\left(1 - \frac{K}{h^2}\right)} \, v\right],$$

and this corresponds to what Newton has proved in the forty-fourth Proposition of the ninth Section.

The analytical formulæ that have been deduced, when translated into the language of curves, correspond exactly to the results obtained by Newton in his ninth Section: but they are deduced from cases entirely fictitious. The disturbing forces which act in Nature do not act solely in the direction of the radius; and, since this is Newton's supposition in the above-mentioned Section, the Propositions therein contained cannot explain completely the Planetary Phenomena.

One of the most noted of those phenomena, is the progression of the Lunar Apogee; and, probably with a view to its explanation, Newton originally constructed the ninth Section: a section, more than any other, abounding with curious, novel, and refined methods.

It is true, that a disturbing force acting solely in the direction of the radius, such as has been supposed in the preceding instances, will cause a *progression* of the apogee; and, it is evident, would, by assuming the Sun's disturbing force of a convenient magnitude, give, as a result of calculation, the just value of that progression. This, however, is not to explain the phenomenon; since, in order to obtain the above-mentioned *value*, it is necessary to assume the Sun's disturbing force nearly the double of what it really is.

The chief merit, then, of that ingenious section (the 9th) of the Principia, consists in the idea of a moveable ellipse. To this (we may conjecture) Newton was led, by Kepler's discoveries and his own investigations, which established the nearly elliptical forms of the orbits of the planets, and by the results of Astronomical observations which shewed the Aphelia of those orbits to be progressive.

There have been mathematicians, however, who have wished to discover in that section more than Newton meant it should contain, and have dispensed with the *tangential* disturbing force, although its operation is as certain as that of the disturbing force which acts in the direction of the radius. And this is strange, since there are no probable or paramount arguments, by which it can be made to appear that, in the investigation of the *progression* of the Lunar Apogee, a right result is to be looked for, when one source of that inequality is rescinded.

Newton, it is true, no where affirms that the progression cannot be determined by the principles, and according to the method of the ninth Section; nor, as it is known, has he given a solution of that problem. He says, Scholium, Prop. xxxv. ed. 1. that he had found by calculation, the quantity of the progression; but, the method either did not completely satisfy him, or did not harmonize * with the *stile* of his other investigations.

The question of the progression of the Lunar Apogee, and the analytical method of determining its quantity, will be resumed in another part of this Treatise. We must now regain the direct course of investigation; and, as it has been already suggested, the next attempts ought to be directed towards the conversion of Π into a series of cosines, such as A cos. $m v$ +

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* The relative *beauty* and *accuracy* of the geometrical and analytical methods is a point not easily decided on. But, their relative *power and efficiency* may be estimated. Physical Astronomy presents to us various cases, in which the analytical method has succeeded in affording true results, whilst the geometrical has failed. The one in the text, the progression of the Lunar Apogee, has never been determined by the latter method.

B cos. nv + &c. Instead, however, of attempting that on a general scale, we prefer (with a view to the interests of the Student) to proceed by instances: beginning with the most simple, and passing on to others that become more complex by the largeness of the disturbing force, and by the obliquity of the direction of its action to that of the centripetal force.

CHAP. VIII.

First Solution of the Problem of the Three Bodies under its most simple Conditions: that is, when the Body, previously to the Action of the Disturbing Force, is supposed to revolve in an Orbit without Eccentricity and Inclination; the Orbit, changed by the Action of the Disturbing Force, not strictly Elliptical.

THE instances in the preceding Chapter were intended principally to explain the cause of that introduction of the arcs of a circle which renders faulty the expression of the radius vector. They have served that end, and the purpose of illustration, as well as more complex instances would have done. But they are altogether fictitious and hypothetical, since they exclude, besides other conditions, the essential one of a tangential disturbing force.

The results of the ninth Section of the *Principia* of Newton have been compared and made to correspond with certain peculiar integral values of the differential equation (see pp. 109, 121.). In both cases, there is the same supposition with regard to the disturbing force. In the ninth Section, Newton's *Extraneous* Force, as it is there called, acts solely in the direction of the radius: and the disturbing force has been expounded by this equation,

$$\Omega = \frac{K}{h^2 u^n}.$$

Two cases have been considered, when $n = -1$, and when $n=3$, (see pp. 109, 121.) that is, when the disturbing force varies inversely as the cube of the distance, and when A varies as the distance. In the former, the *exact* value of u is expressed by an equation, such as

$$u = \frac{1}{L} + \frac{e}{L} \cos. c v,$$

whatever e be; or, [which is Newton's mode of considering the subject, (see Prop. XLIV.)] the body's place can always be found, and *exactly*, in a moveable ellipse, whatever its eccentricity be. In the latter case, when the disturbing force varies as the distance, an equation, such as

$$u = \frac{1}{L} + \frac{e}{L} \cos. c v,$$

approximately represents the value of u; and that, only when e is very small: or, according to Newton, (ninth Section, Prop. XLV.) the body's place may *nearly* be found in a moveable ellipse, when the orbit's eccentricity is very small; and the like equations and constructions obtain approximately for all other values of n*.

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* It has been already remarked, (p. 122.) that some mathematicians, persuaded that Newton meant to find the progression of the Lunar Apogee by the method of the ninth Section, have pursued that method. Now in that Section there is no tangential disturbing force, and, besides, the expression for that part of the Sun's disturbing force which acts in the direction of the radius, (see p. 60.) is unlike Newton's $\left(\frac{1}{A^2} + c A\right)$. It was necessary for them, therefore, to shew by some probable arguments, that, in a problem of such importance as that of the Lunar Apogee, the former force could be dispensed with, and that the latter might be reduced to Newton's form.

Now, with regard to the first point; the tangential force T (see p. 60.) is $\frac{3 m' r}{2 r'^3} \sin. 2\omega$, which, from the largeness of the denominator, is very small. But, besides its smallness, its effects counteract each other: since, if $\frac{3 m' r}{2 r'^3} \sin. 2\omega$ accelerate the body, $\frac{3 m' r}{2 r'^3} \sin. (180 + 2\omega) = - \frac{3 m' r}{2 r'^3} \sin. 2\omega$ equally retards the body; which counteraction (since ω may be any angle) must accordingly take place for all corresponding points of the orbit. The mean effect therefore of this force, it was presumed, It has been just remarked that the instances of the preceding Chapter, framed for the purpose of illustration, are, with reference to the real circumstances in nature, fictitious and hypothetical. But, we may add to this remark, every instance which can be given is, to a certain degree, hypothetical. The inefficiency of the art of calculation obliges us to suppose a greater simplicity in the conditions of our problems than exists. The kind of simplification, however, which will be given to the succeeding instances is different from that which the preceding possess. Instead of excluding altogether the tangential force, its

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sumed, (not rightly inferred) would not materially affect the progression of the apsides.

With regard to the second point, we have, by p. 60.

$$P = \frac{M + m}{r^2} - \frac{m' r}{2 r'^3} - \frac{3 m' r}{2 r'^3} \cos. 2 \omega.$$

If we substitute in the last term $180 \mp 2 \omega$ instead of 2ω, it becomes (see *Trig.* p. 28.)

$$\frac{-3 m' r}{2 r'^3} \cos. (180 \mp 2 \omega) = \frac{3 m' r}{2 r'^3} . \cos. 2 \omega,$$

the value of P, therefore, is as much increased by the last term, in this situation of the body, as it was diminished in the former, and, since the same holds whatever ω be, that is, since the same result is true for every point in the orbit, the last term $\left(\frac{3 m' r}{2 r'^3} \cos. 2 \omega\right)$ is said, during an entire revolution, to be *destroyed by the opposition of signs.* Under this explanation, then, the *mean* force may be said to be

$$\frac{M + m}{r^2} - \frac{m' r}{2 r'^3}, \text{ or, } \mu u^2 - \frac{m'}{2 r'^3 u},$$

and, if that force alone operates, the equation would be (see p. 111.),

$$u = \left(\mu - \frac{4 m'}{2 r'^3}\right)^2 (1 + e' \cos. c v),$$

in which $c = \sqrt{\left(1 - \frac{3 m'^4}{2 r'^3}\right)} . (h^2 = 1)$,

the progression of the Apogee being accordingly

$$1 - c = 1 - \sqrt{\left(1 - \frac{3 m'}{2 r'^3}\right)}$$

approximate value will be assumed and substituted in Π : and, of the force that acts in the direction of the radius, all the essential terms at least, will be retained, although in determining their coefficients many small quantities will be rejected. And thus it shall happen, that the results will not be altogether remote from the truth, but will accord, in some degree, with the observed phenomena.

If we look to the History of the Problem of the Three Bodies, it exhibits a series of solutions successively more and more exact. The Calculus, which was the instrument of solution, grew up with Physical Astronomy, and, as it advanced, additional conditions were introduced into the problem; so that, as the fruit of time, Laplace's Theory of the Moon, (without any reference to the genius of the two authors) is necessarily more perfect than Clairaut's.

The present business of this Treatise, however, is not with the most complete solutions. Intended to serve as an introduction to Physical Astronomy, it will begin with the most simple cases, and be guided, very nearly, by their historical order.

But, even according to this plan, there are two ways of proceeding. We may either select what are, in fact, the most simple cases in Nature, or we may, by hypothesis, simplify the conditions of some of the more complex cases. For instance, Venus revolving in a nearly circular orbit, and disturbed by a body as remote as Saturn, the plane of whose orbit is very little inclined to that of Venus's, is nearly as simple a case as is that of the Moon, when, as in the first essays of solution, that planet is supposed to revolve in an orbit circular, and coincident with the plane of the ecliptic. This regards the analytical difficulty of solution; but, with reference to arithmetical exactness, it is plain that the results in the first case, when specific numbers are substituted, must be more conformable to observation than those in the latter.

We will now proceed to a series of solutions of the *Problem of the Three Bodies*, or, in the analytical mode of considering the subject, to a series of integrations of the differential equation,

$$\frac{d^2 u}{d v^2} + u + \Pi = 0,$$

and of the other two equations of p. 95.

EXAMPLE 1.

It is required to find the value of the inverse of the radius vector $\left(u = \frac{1}{\rho}\right)$ when the body, revolving in a circular orbit round an attracting centre or body, is *disturbed* by the action of a very remote body, which revolves also in a circular orbit, the plane of which is coincident with the plane of the other orbit. The law of the force, whether it be centripetal or disturbing, is supposed to vary according to the inverse square of the distance.

The first operation will be to find the value of Π, and (see pp. 100, 104, &c.) to convert it into a series of cosines of multiples of the arc v.

Value of Π.

$$\Pi \text{ (see p. 95.)} = \frac{T \,.\, \frac{d u}{d v} - P u}{u^3 \left(h^2 + 2 \int T \frac{d v}{u^3}\right)},$$

$$= - \frac{P}{h^2 u^2 \left(1 + \frac{2}{h^2} \int T \frac{d v}{u^3}\right)},$$

$$\left(\text{since } \frac{d u}{d v} = 0\right),$$

$$= - \frac{P}{h^2 u^2} \left(1 - \frac{2}{h^2} \int \frac{T d v}{u^3}\right) \text{ nearly.}$$

This is a very simple expression for Π, obtained on two suppositions: the first, (which does not strictly hold of any case in Nature,) is, that the orbit is circular, and consequently that u is constant, and $\frac{d u}{d v} = 0$: the second, which is nearly true in every instance in the planetary system, namely, that the disturbing force is very small, and consequently that the square of the term $\int \frac{T\,dv}{u^3}$ compounded of it, may, in the expansion of the denominator of Π, be rejected.

Values of P *and* T.

By page 60,

$$P = \mu u^2 - \frac{m' u'^3}{2u} - \frac{3 m' u'^3}{2u} \cos. 2\omega$$

$$= \frac{\mu}{a^2} - \frac{m' a}{2a'^3} - \frac{3 m' a}{2a'^3} \cos. 2\omega,$$

$$\text{and } T = -\frac{3 m' u'^3}{2u} \sin. 2\omega$$

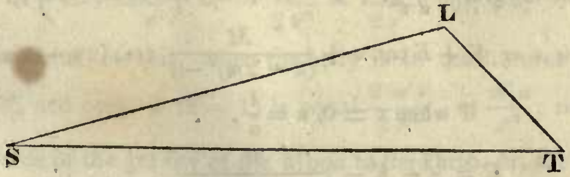
$$= -\frac{3 m' a}{2a'^3} \sin. 2\omega,$$

a and a' being respectively equal to $\frac{1}{u}$ and $\frac{1}{u'}$.

Now, since u is, by hypothesis, constant, the sole thing that remains to be done, in determining the value of Π, is to find the values of $\cos. 2\omega$ and of $\int \sin. 2\omega \,.\, dv$: and this can be done, if we can express 2ω in terms of v.

Values of cos. 2ω, *and sin.* 2ω.

The angle ω, or LTS, is the difference of v and v'. Now v and v', when the orbits are circular, are proportional to the mean



motions of L and T': let those mean motions be n, n'; then

$$\omega = v - v,$$

$$= v - \frac{n' v}{n}$$

$$= v - m v \quad \left(m = \frac{n'}{n}\right);$$

$$\therefore \ \sin. 2\omega = \sin. (2v - 2mv),$$
$$\cos. 2\omega = \cos. (2v - 2mv).$$

Values of $\int \frac{T\, d v}{u^3}$.

$$\int \frac{T d v}{u^3} = - \int \frac{3 m' u'^3}{2 u^4} \sin. (2v - 2mv) . d v$$
$$= \frac{3 m' a^4}{2 a'^3} . \frac{\cos. (2v - 2mv)}{2 - 2m}:$$

(*Trig.* p. 98.)

Hence, by substituting the preceding values,

$$\Pi = - \frac{\mu}{h^2} + \frac{m'}{2h^2} \frac{a^3}{a'^3}$$
$$+ \left(\frac{3m'}{2h^2} \frac{a^3}{a'^3} + \frac{3 \mu m' a^4}{(2 - 2m) h^4 a'^3}\right) . \cos. (2v - 2mv),$$

in which expression the terms involving $\left(\frac{m' a^3}{a'^3}\right)^2$ are neglected.

In the above value of Π, the only variable term is a cosine of a multiple of the arc v: therefore, (see p. 104.) the differential equation can be integrated, and

$$* \ u = \frac{\mu}{h^2} - \frac{1}{2h^2} . \frac{m' a^3}{a'^3}$$
$$+ E \cos. v$$

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* The integral of $\frac{d^2 u}{d v^2} + u - K + M . \cos. (2v - 2mv)$,

$$\text{is } u = K + E \cos. v + \frac{M}{(2 - 2m)^2 - 1} \cos. (2v - 2mv),$$

if when $v = 0, u = \frac{1}{a}$,

$$E = \frac{1}{a} - K - \frac{M}{(2 - 2m)^2 - 1}$$

$$= \text{(in the present case) } \frac{1}{2h^2} . \frac{m' a^3}{a'^3} - \frac{M}{(2 - 2m)^2 - 1},$$

since $\frac{\mu}{h^2} = \frac{1}{a} = \frac{1}{a}$, nearly.

$$+ \frac{3\,m'\,a^3}{2\,h^2\,a'^3\,[(2-2\,m)^2-1]}\left(1+\frac{\mu}{1-m}\,\frac{a}{h^2}\right)\cos.\,(2\,v-2\,m\,v),$$

which is the first approximate value of u, and shews, that the effect of the disturbing force is to render u variable, or, in other words, to destroy the circular form of the orbit.

We may give a different form to the preceding expression by substituting for the values of h^2, and of $\frac{m'\,a^3}{a'^3}$.

Since, by hypothesis, the disturbing force T is very small, the two expressions for the element of the time (see pp. 95, 96.) in the elliptical and disturbed systems will be nearly equal: the value of h, therefore, will be nearly the same in both. But, in the former, that value (see p. 25.) is equal $\sqrt{\mu a}\,.\,(1-e^2)$, a representing the mean distance. If the system, however, be disturbed by the action of a third body, the constant distance which corresponds to a will be changed: consequently, if we choose to retain the symbol a still to denote this latter distance, we must express the former mean distance by some other symbol: suppose it to be $a_{,}$. Hence we have, in the case of three bodies,

$$\begin{aligned} h^2 &= \mu\,a_{,}\;(1-e^2) \\ &= \mu\,a_{,} \end{aligned}$$

in the present instance in which $e = 0$.

With regard to the second point, the value of $\frac{m'\,a^3}{a'^3}$, if we refer to p. 109, we shall find that

$$\Omega\left(=P-\frac{\mu}{r^2}\right) = -\,\frac{m'\,r}{2\,r'^3} - \frac{2\,m'\,r}{2\,r'}\;\cos.\,2\,\omega,$$

and the value of this, when the body is in quadratures (when $\omega = 90^\circ$, and $\cos.\,2\,\omega = -\,1$) is equal $\frac{2\,m'\,r}{2\,r'^3}$, or $\frac{m'\,a}{a'^3}$; now the mean force of the gravity of the Moon to the Earth, or, the centripetal force is represented by $\frac{\mu}{a^2}$: if therefore, we represent this latter force by 1, $\frac{m'\,a}{a'^3}\times\frac{a^2}{\mu}$, or $\frac{m'\,a^3}{\mu\,a'^3}$ will represent the disturbing force in its mean value; this will be hereafter denoted by the symbol K.

In specific cases the arithmetical value of the disturbing force, in its mean value, is easily expounded: for instance, in the case of the Moon, m' denoting the mass of the Sun.

The force by which the Earth is drawn towards the Sun $= \frac{m'}{a'^2}$.

But this same force, by Newton, Prop. 4. and p. 29. $= \frac{a'}{(\text{period } \oplus)^2}$;

$$\therefore \frac{m'}{a'^3} = \frac{1}{(\text{period } \oplus)^2},$$

and similarly, when m denotes the Moon's mass, and M the Earth's,

$$\frac{M+m}{a^3}, \text{ or } \frac{\mu}{a^3} = \frac{1}{(\text{period ☽})^2}.$$

Hence,

$$\frac{m'a^3}{\mu a'^3} = \left(\frac{\text{period ☽}}{\text{period } \oplus}\right)^2 = {}^{*}\left(\frac{27^d\ 7^h\ 43^m\ 4^s}{365^d\ 5^h\ 48^m\ 51^s}\right)^2$$

$$= \left(\frac{27.32158}{365.24226}\right)^2 = (.01748013)^2 = .005595, \text{ nearly},$$

or, if expressed by a vulgar fraction, the mean value of the disturbing force,

$$K, \text{ or } \frac{m'a^3}{\mu a'^3} \left[= \left(\frac{n'}{n}\right)^2 = m^2 \right] = \frac{1}{178}, \text{ nearly},$$

when the mean gravitation of the Moon to the Earth is represented by 1.

If we now substitute the preceding values for h^2 and $\frac{m'a^3}{\mu a'^3}$, in the expression for u, (see p. 130.), we shall have

$$u = \frac{1}{a} - \frac{K}{2a}$$

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* *Astronomy*, pp. 305, 306. See Newton, Lib. III. Prop. xxv.

$$+ E \cos. v$$

$$+ \frac{3K}{2a_{,}[(2-2m)^2-1]}\left(1 + \frac{1}{1-m}\,\frac{a}{a_{,}}\right).\cos.(2v-2mv),$$

where

$$E = \frac{K}{2a_{,}} - \frac{3K}{2a_{,}[(2-2m)^2-1]}\cdot\frac{2-m}{1-m}, \text{ nearly,}$$

supposing $a = a_{,}$, and u to $= \frac{1}{a}$ when $v = 0$.

This is the first approximate value of u: and it shews us that the former constant radius is, by the action of the disturbing force, rendered variable.

$a_{,}$ is the radius of the circular orbit in which, when the disturbing force is excluded, the body is supposed to revolve: and a is the constant part of the radius vector in the disturbed orbit, and between a and $a_{,}$, when $v = 0$, we have this equation,

$$\frac{1}{a} = \frac{1}{a_{,}} - \frac{K}{2a_{,}}, \text{ whence}$$

$$a = a_{,}\left(1 + \frac{K}{2}\right), \text{ nearly,}$$

$$\text{and } a_{,} = a\left(1 - \frac{K}{2}\right), \text{ nearly.}$$

If, from the above formula, we wish to compute the Moon's radius (supposing that to be the body disturbed) in conjunction, opposition, and quadratures *, we must substitute respectively for v, 0, 180° and 90°.

The equation to an ellipse, (see p. 27.) is of this form

$$u = \frac{1}{p} + \frac{e}{p}\cos. v;$$

but the preceding value of u is of the form

$$u = \frac{1}{a_{,}} - \frac{K}{2a_{,}} + L\cos.(2v - 2mv) - E\cos. v,$$

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* See *Astronomy*, pp. 43, 44.

it differs therefore from an ellipse, on account of the term $L \cdot \cos.(2v - 2mv)$. The effect of the disturbing force, then, inasmuch as we are able to infer from the preceding deduced value of u, is *not* to change the circular orbit into an elliptical.

But the inference from that deduced value of u may not, with reference to the real change in the form of the orbit, be strictly true, since that value is only the first approximate one. To ascertain, therefore, the justness of the inference, we ought to deduce a second value of u by substituting the one just obtained in Π, and by again integrating the differential equation. But it is easy to perceive, without going through this process, that its result will be, not to rescind terms like $L \cos.(2v - 2mv)$, but to augment the value of u by new terms involving the cosines of new arcs: so that, the second equation determining u will be still more remote from an equation to an ellipse than the first is.

If we were, however, to substitute the first value of u in Π, the second resulting value would contain an arc of a circle *without the sign*, and be faulty: for, (see p. 95.) one term in Π is $-\frac{P}{h^2 u^2}$, and since (see p. 129.) the middle term of P is $-\frac{m' u'^3}{2u}$, Π will contain this term $\frac{m' u'^3}{2h^2 u^3}$: now, since (see p. 132.)

$$u = \frac{1}{a_\backslash} - \frac{K}{2a_\backslash} - E \cos.v + \&c.$$

$\frac{m' u'^3}{2h^2 u^3}$, when expanded, must contain a term such as $N \cos.v$; therefore Π must, and if Π contains such a term, then (see pp. 107, &c.), the process of integration will necessarily introduce, into the value of u, this term $-\frac{N}{2} v \cdot \sin.v$. We must, therefore, according to the rules laid down in pp. 110, &c. not directly pursue the plain method of approximation, but deviate from it in order to avoid its difficulties.

The conditions of the preceding case have been assumed the most simple possible, in order to procure an easy introduction to the solution of the differential equation. The solution that has been obtained gives us merely an imperfect value of u, which, since it represents the inverse of the radius, is proportional (see *Astronomy*, pp. 95, &c.) to the *parallax.* The deduction of the value of this quantity is one use then, of the preceding integration: but it is not the chief use: that is to be found in the means afforded us of deducing the longitude (v), which depends, in the first instance, on $n t$ the mean longitude: but t, in order to be determined, requires that u should previously be known, since (see p. 95.)

$$d t = \frac{d v}{h u^2 \sqrt{\left(1 + \frac{2}{h^2} \int \frac{T d v}{u^3}\right)}},$$

the integral of which cannot be correctly* found, as it is plain, except we know u.

The particular method of forming the several terms that represent the true longitude, will be explained in a future part of this Treatise. The present concern is with the differential equation, on which the value of u depends. We shall endeavour to obtain that value by a series of successive corrections.

The value of the inverse of the radius of the orbit, from being constant, becomes, by the agency of the disturbing force, (and by the process of one approximation and integration) of this form:

$$u = \frac{1}{a_{\backslash}} - \frac{K}{2a_{\backslash}} - E \cos. v + L \cos. (2 v - 2 m v),$$

and this, as it has been observed, is not the equation to an ellipse: it would be, were the last term rescinded. That last term expounds what in Astronomical language is called an *Inequality:* the *argument* is the arc $2 v - 2 m v$: and the *coefficient* is

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* By this term it is not meant to be understood that, if a correct value of u should be obtained, the integral of the differential can be expressed by a definite equation. It can only be expressed by a series: and, by a reversion of that series, v must be expressed in terms of t.

$$L = \frac{3K}{2a_{,}}\left(1 + \frac{1}{1-m}\,\frac{a}{a_{,}}\right) \times \frac{1}{(2-2m)^2 - 1}\ *.$$

The term E cos. v (see *Astron.* pp. 322, 324.) expounds what is called the *Elliptic Inequality.* The rule therefore, for finding u may be expressed either by saying that we must correct its elliptic value by means of the *equation* represented by L cos. $(2v - 2mv)$: or, that we must correct its constant value by means of two equations, one due to the elliptic inequality, the other to that inequality of which the argument is $2v - 2mv$.

The instance that has been given in this Chapter is one of the most simple that can be imagined when no essential condition is excluded. It will not exactly suit any case in nature: not even that of Venus disturbed by the action of Saturn: still less that of the Moon disturbed by the Sun. It must fail to represent this latter case for several reasons, of which the most prominent are, the eccentricities and inclinations of the Solar and Lunar orbits. Still, on the preceding solution, as on a basis, may more correct ones be founded: by introducing new conditions and by applying corrections proportional to them to the values of P, T, &c. But the method will be best understood by the Example of the succeeding Chapter.

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* See *Astronomy,* pp. 324, &c.

CHAP. IX.

Continuation of the Solution of the Problem of the Three Bodies: the Orbit of the disturbed Body is supposed to be Elliptical: the resulting Value of the Radius Vector thereby augmented with additional Terms. Clairaut's First Method of determining the Progression of the Lunar Apogee.

EXAMPLE 2.

IT is required to find the inverse of the radius vector $\left(u = \frac{1}{\rho}\right)$, when the body, revolving in an elliptical orbit of very small eccentricity, is disturbed by the action of a very remote body which revolves in a circular orbit, the plane of which is coincident with that of the elliptical orbit.

Into this Example, only one (see Ex. 1. p. 128.) new condition is introduced, namely, the eccentricity of the disturbed orbit, and that is supposed to be very small.

Value of Π. (*See pp.* 128, &c.).

$$\Pi = \frac{T\frac{du}{dv} - Pu}{h^2u^3\left(1 + \frac{2}{h^2}\int T\frac{dv}{u^2}\right)}$$

$$= \frac{1}{h^2u^3}\left(T\frac{du}{dv} - Pu\right)\left(1 - \frac{2}{h^2}\int\frac{Tdv}{u^3}\right)$$

$$= \frac{1}{h^2u^3}\cdot T\frac{du}{dv} - \frac{P}{h^2u^2} + \frac{2P}{h^4u^2}\int\frac{Tdv}{u^3},$$

neglecting the product of the first and last terms of the two factors.

Now, see p. 60. and making $\mu = 1$,

$$\frac{P}{u^2} = 1 - \frac{m' u'^3}{2 u^3} - \frac{4 m' u'^3}{2 u^3} \cos. 2\,\omega\,;$$

therefore, the second and third terms of $\frac{2P}{h^4 u^2} \int \frac{T\,dv}{u^3}$, would involve (since $\frac{m' u'^3}{u^3}$, and T are of the order of the disturbing forces) the square of the disturbing force: if we neglect them, by reason of their minuteness, and retain solely the first term, then

$$\frac{2P}{h^4 u^2} \int \frac{T\,dv}{u^3} = \frac{2}{h^4} \int \frac{T\,dv}{u^3},$$

and

$$\Pi = \frac{1}{h^2 u^3} \cdot T \frac{du}{dv} - \frac{P}{h^2 u^2} + \frac{2}{h^4} \int \frac{T\,dv}{u^3}.$$

This value of Π differs from the former, (see p. 128.) by the first term, which is here retained; since, when $\frac{1}{u}$ is not constant but the radius vector of an ellipse, $\frac{du}{dv}$ is not equal 0. The other condition, that of the smallness of the disturbing force, is the same in both cases, and on that account, in expanding the denominator of Π, there is the same rejection of the terms that involve the square and higher powers of $\int T \frac{dv}{u^3}$.

But we may obtain an expression without expanding the denominator of Π, and, by merely multiplying every term of

$$\frac{d^2 u}{dv^2} + u + \Pi = 0,$$

by $\left(1 + \frac{2}{h^2} \int \frac{T\,dv}{u^3}\right)$: the differential equation then becomes

$$\frac{d^2 u}{dv^2} + u + \left(\frac{d^2 u}{dv^2} + u\right) \frac{2}{h^2} \int \frac{T\,dv}{u^3} + \frac{1}{h^2 u^3} T \frac{du}{dv} - \frac{1}{h^2} \frac{P}{u^2} = 0,$$

and it makes very little difference in the result, whether we use this, or that form of the differential equation which arises on subtituting instead of Π its former value (l. 9.).

Value of T, (*see p.* 129.).

The expressions for T and P, when sin. $2\,\omega$ and cos. $2\,\omega$, are merely signified and not expanded, are the same in this instance as in the preceding values of cos. $2\,\omega$ and sin. $2\,\omega$ (see p. 129.).

Values of cos. $2\,\omega$, *and sin.* $2\,\omega$, (*see p.* 129.)

The orbit of the revolving body being now elliptical instead of circular, the angle v, instead of representing, as before, indifferently the true and mean anomaly, will now solely represent the former. v', however, (see p. 137.) is the same as in the former Example, and constant. The process for finding it in terms of v (for this is necessary) will be longer and more difficult * than the previous one of p. 129; but it will begin and be instituted from the same principle; and this principle consists in equating the two expressions for the time, one belonging to the interior and revolving body, the other to the remote and disturbing body.

To find the time in the first case

$$dt = \frac{dv}{hu^2}, \text{ nearly,}$$

$$= \frac{dv}{\sqrt{(a_{,}\,.\,(1-e^2)}} \times \frac{a^2\,.\,(1-e^2)}{(1+e\cos.\,cv)^2},$$

in which

$$h = \sqrt{[a_{,}\,(1-e^2)]},$$

$$u = \frac{1+e\,.\cos.\,cv}{a\,.\,(1-e^2)},$$

$a_{,}$ being the mean distance, such as it would be were there no perturbation; $\frac{1}{a\,.\,(1-e^2)}$, or nearly, $\frac{1}{a}(1+e^2)$ representing the constant part of u; and c being such a quantity that $(1-c)v$ denotes the progression of the apogee, (see pp. 121, &c.).

Now, if we expand the expression for dt, and neglect the terms that involve e^2 and higher powers of e, we shall have

$$dt = \frac{a^2}{\sqrt{a_{,}}}\,dv\,(1-2e\,.\cos.\,cv),$$

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* 'Il faut quelque preparation pour reduire les valeurs des sinus et cosinus de $h\,.\sin.\,mv$.' Clairaut, *Mem. Acad.* 1754. p. 533.

whence $t = \frac{a^2}{\sqrt{a_{,}}}\left(v - \frac{2e}{c}\,\text{sin.}\,cv\right)$

$$= a^{\frac{3}{2}}(v - 2e\,\text{sin.}\,cv),$$

very nearly, since a is nearly $= a_{,}$, and c, nearly, $= 1$. To find the time in the circular orbit,

$$t = \frac{1}{\sqrt{a'}}\int\frac{dv'}{u'^2} = \frac{v'}{\sqrt{a'}} \times a'^2 = a'^{\frac{3}{2}}\,v'.$$

Hence, making $n = a^{-\frac{3}{2}}$, and $n' = a'^{-\frac{3}{2}}$, and $m = \frac{n'}{n}$, we have, by equating the two expressions for the time,

$$v' = mv - 2em\,\text{sin.}\,cv,$$

and $\omega\,(= v - v') = v(1 - m) + 2em\,\text{sin.}\,cv$.

Hence, by a peculiar computation (see *Trig.* ed. 2. p. 103.) cos. $2\,\omega$, sin. $2\,\omega$ may be approximately found,

$$\text{cos.}\,2\,\omega = \text{cos.}\,(2v - 2mv) - 2me\,.\,\text{cos.}\,(2v - 2mv - cv)$$
$$+ 2me\,.\,\text{cos.}\,(2v - 2mv + cv),$$

$$\text{sin.}\,2\,\omega = \text{sin.}\,(2v - 2mv) - 2me\,.\,\text{sin.}\,(2v - 2mv - cv)$$
$$+ 2me\,.\,\text{sin.}\,(2v - 2mv + cv).$$

and these are formulæ, which, in the analytical mode of treating the problem of the three bodies and the lunar theory, cannot, it would seem, be essentially dispensed with; since they have been deduced by all writers on this subject, from Clairaut their first inventor, to Laplace the latest author on Physical Astronomy*.

In the expression for dt, (see p. 139. l. 16.) instead of $\frac{1 + e\,\text{cos.}\,v}{a(1 - e^2)}$, $\frac{1 + e\,\text{cos.}\,cv}{a\,.\,(1 - e^2)}$, has been substituted for the value of u. For if the former were substituted, then, as it has been explained, (pp. 107. &c. 134.) there would result a faulty expression for

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* See Clairaut, *Mem. Acad. Paris,* 1745, p. 348: *Theorie de la Lune,* p. 20. Simpson's *Tracts,* p. 171. Mayer's *Theoria Lunæ,* pp. 15. 17. Dalembert's *Theorie de la Lune.* Laplace, *Mec. Celeste,* Tom. II. pp. 189, &c.

u involving arcs of a circle *. The substitution of the latter value of u, as its truer value, is equivalent, when we speak of curves, to

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* The expression for u would involve arcs of circles, because (see pp. 107, &c.) the differential equation

$$\frac{d^2 u}{d v^2} + u + \Pi = 0,$$

the first approximate value of Π would contain a term such as $\pm A \cos. v$ from the substitution of $\frac{1}{p} + \frac{e}{p} \cos. v$ for u. But if Π contains $A \cos. v$, then, since $\pm A \cos. v = \pm \frac{A}{e} (up - 1)$, the preceding equation under a different transformation contains a term such as $\pm \frac{Ap}{e} u$, and, consequently, the differential equation might be thus expressed,

$$\frac{d^2 u}{d v^2} + \left(1 \pm \frac{Ap}{e}\right) u + \&c. = 0,$$

and the integral value of u in this equation, (see p. 101.) is of the form

$$u = H + L \cos. \sqrt{\left(1 \pm \frac{Ap}{e}\right)} u + \&c.$$

and at this equation Dalembert, (see *Theorie de la Lune,* p. 40.) in his first approximation arrived, and Clairaut, guided either by analytical investigation, or (as it would seem by a passage in the following Extract) by the results of Astronomical observation, soon discovered that an equation such as $r = \frac{p}{1 - \cos. m U}$ more adequately represented the radius vector than the common elliptical equation. 'Il faut donc choisir pour premiere equation de l'orbite lunaire, quelqu' equation qui ne s'ecarte jamais considerablement de la vraie. Pour faire ce choix, je remarque qu'au lieu de l'equation $\frac{p}{r} = 1 - c \cos. U$, qui exprime l'ellipse primitive, si on prend $\frac{p}{r} = 1 - c \cos. m U$, ou aura l'equation d'une courbe formeé en faisant mouvoir une ellipse autour de son foyer, en telle sorte que son apside decrive un angle qui soit a celui que la planete parcourt dans cette ellipse, comme $1 - m$ ad 1 : et j'en conclus qu'en se rapportant au moins a ce que les *observations nous apprenent,* cette equation doit etre plus voisine de celle qui exprime veritablement l'orbite : que la seule equation $\frac{p}{r} = 1 - c \cos. U$, pourvu que la lettre m soit determinê convenablement. *Mem. Acad.* 1745, p. 346.

this statement, namely, that the body's place is more nearly in the circumference of a moveable, than of a fixed ellipse.

Value of $\frac{1}{h^2}\,\frac{T}{u^3}\,\frac{du}{dv}$, (*see p.* 130.)

In the former case, since the orbit was supposed to be originally circular, $\frac{du}{du}$ was equal 0, and consequently the preceding term was 0. In the present case,

$$u = \frac{1 + e \cos. cv}{a(1 - e^2)} = \frac{1}{a} + \frac{e}{a} \cos. cv, \text{ nearly};$$

$$\therefore \frac{du}{dv} = -\frac{ce}{a} \sin. cv.$$

Now, rejecting the terms that involve the square and higher powers of e, we have (see p. 139.)

$$\frac{1}{h^2} = \frac{1}{a_{,}},$$

$$\frac{T}{u^3} = -\frac{3m'}{2}\,\frac{u'^3}{u^4} \sin. 2\omega$$

$$= -\frac{3m'}{2}\,\frac{a^4}{a'^3}(1 - 4e \cos. cv) \sin. 2\omega.$$

Hence, making as before in p. 132. $K = \frac{m'a^3}{a'^3}$,

$$\frac{1}{h^2}\,\frac{T}{u^3}\,\frac{du}{dv} = \frac{3K}{2a_{,}} ce \sin. cv \,.\, \sin. 2\omega (1 - 4e \,.\, \cos. cv)$$

$$= \frac{3K}{2a_{,}} ce \sin. cv \,.\, \sin. 2\omega,$$ (rejecting the term involving e^2)

$$= \frac{3K}{2a_{,}} ce \,.\, \sin. cv \,.\, \sin. (2v - 2mv)$$

$$= \frac{3K}{4a_{,}} ce [\cos. (2v - 2mv - cv) - \cos. (2v - 2mv + cv)].$$

(see *Trig.* p. 26. formula [*e*]).

Value of $\frac{\mathrm{P}}{\mathrm{h}^2\mathrm{u}^2}$, *(see pp.* 129. 138.)

$$\frac{P}{h^2u^2} = \frac{1}{h^2} - \frac{m'u'^3}{2h^2u^3} - \frac{3m'u'^3}{2u^3}\cos.\,2\omega$$

$$= \frac{1}{a_,} - \frac{K}{2a_,}(1+e\,.\cos.\,cv)^{-3} - \frac{3K}{2}(1+e\cos.\,cv)^{-3}\cos.\,2\omega,$$

Now

$(1+e\cos.\,cv)^{-3} = 1 - 3e\cos.\,cv$, nearly, and

$$\begin{aligned}(1-3e\cos.\,cv)\cos.\,2\omega = \cos.\,(2v-2mv) - 2me\,.\cos.\,(2v-2mv-cv),\\ +2me\,.\cos.\,(2v-2mv+cv)\end{aligned}$$

$$\frac{-3e}{2}[\cos.\,(2v-2mv-cv)+\cos.\,(2v-2mv+cv)].$$

Hence, collecting the coefficients of the cosines of the same arcs,

$$\frac{P}{h^2u^2} =$$

$$\frac{1}{a_,} - \frac{K}{2a_,} + \frac{3Ke}{2a_,}.\cos.\,cv - \frac{3K}{2a_,}\cos.\,(2v-2mv)$$

$$+\frac{3Ke}{2a_,}\left[\frac{3+4m}{2}\cos.\,(2v-2mv-cv) + \frac{3-4m}{2}\cos.\,(2v-2mv+cv)\right].$$

Value of $\frac{2}{\mathrm{h}^4}\int\frac{\mathrm{T\,d\,v}}{\mathrm{u}^3}$.

$$\frac{T}{u^3} = -\frac{3m'}{2}\,\frac{u'^3}{u^4}\sin.\,2\omega$$

$$= -\frac{3Ka}{2}(1-4e\cos.\,cv)\left\{\begin{array}{l}\sin.\,(2v-2mv)-2me\sin.\,(2v-2mv-cv)\\ \qquad\qquad +2me\sin.\,(2v-2mv+cv).\end{array}\right\}$$

If these quantities be involved and then expanded, according to the rules of Trigonometry, we shall have (neglecting as before the terms that involve e^2 and the higher powers of e)

$$\frac{T}{u^3} = -\frac{3Ka}{2}\left\{\begin{array}{l}\sin.\,(2v-2mv)-2e(1+m)\sin.\,(2v-2mv-cv)\\ \qquad\qquad -2e(1-m)\sin.\,(2v-2mv+cv)\end{array}\right\}.$$

Hence,

$$\frac{2}{h^4}\int\frac{T\,d\,v}{u^3} =$$

$$\frac{3\,K\,a}{a_{,}^2}\left\{\begin{array}{l}\dfrac{\cos.\,(2\,v-2mv)}{2-2\,m} - \dfrac{2\,(1+m)}{2-2m-c}\,e\,.\cos.\,(2\,v-2\,m\,v-c\,v) \\ \qquad\qquad\qquad - \dfrac{2\,.\,(1-m)}{2-2m+c}\,e\,.\cos.\,(2\,v-2\,m\,v+c\,v)\end{array}\right\},$$

and the value of $\frac{2}{h^2}\int\frac{T\,d\,v}{u^3}$, which is required in the second formula of the equation (see p. 138.) is immediately had by multiplying the preceding value into $h^2 = a_{,}$.

If we use this latter form of the differential equation, then, from the assumed value of u, we must find (see p. 138.),

$$\frac{d^2\,u}{d\,v^2} + u.$$

Now

$$u = \frac{1}{a}\,(1 + e\cos.\,c\,v),\ \text{nearly};$$

$$\therefore\ \frac{d^2\,u}{d\,v^2} + u = \frac{1}{a} + \frac{e}{a}\,(1-c^2)\,.\cos.\,c\,v,$$

and, since it is intended to neglect all terms that involve e^2 and the higher powers of e, we shall have

$$\left(\frac{d^2\,u}{d\,v^2} + u\right)\frac{2}{h^2}\int\frac{T\,d\,v}{u^3} = \frac{3\,K}{2\,a_{,}}\ \frac{\cos.\,(2\,v-2\,m\,v)}{1-m}$$

$$+\ \frac{3\,K\,e\,(1-c^2)}{4\,a_{,}\,.\,(1-m)}\left[\cos.\,(2\,v-2\,m\,v-c\,v)+\cos.\,(2v-2\,m\,v+c\,v)\right].$$

If we now collect and arrange the terms of the differential equation, we shall have

$$0 =$$

$$\frac{d^2\,u}{d\,v^2} + u - \frac{1}{a_{,}} + \frac{K}{2\,a_{,}} - \frac{3\,K\,e}{2\,a_{,}}\cos.\,c\,v$$

$$+\ \frac{3\,K}{2\,a_{,}}\,.\left(\frac{2-m}{1-m}\right)\cos.\,(2\,v-2\,m\,v)$$

$$+\frac{3K}{a_{,}}\left(\frac{c}{4}-\frac{3+4m}{4}-\frac{2(1+m)}{2-2m-c}+\frac{1-c^2}{4(1-m)}\right)e\cos.(2v-2mv-cv)$$

$$-\frac{3K}{4a_{,}}\left(c+3-4m+\frac{8(1-m)}{2-2m+c}+\frac{1-c^2}{1-m}\right)e.\cos.(2v-2mv+cv).$$

If we substitute for the coefficients of the cosines of the arcs $2v-2mv$, $2v-2mv-cv$, $2v-2mv+cv$, the letters A, B, C, we shall have, by integrating according to the process of p. 100.

$$\begin{aligned}u=\frac{1}{a_{,}}-\frac{K}{2a_{,}}-\frac{3Ke}{2a_{,}(c^2-1)}\cos.cv\\ +\frac{A}{(2-2m)^2-1}.\cos.(2v-2mv)\\ +\frac{B}{(2-2m-c)^2-1}.\cos.(2v-2mv-cv)\\ +\frac{C}{(2-2m+c)^2-1}.\cos.(2v-2mv+cv)\end{aligned}$$

$$+\left(\frac{3Ke}{2a_{,}(c^2-1)}-\frac{A}{(2-2m)^2-1}-\frac{B}{(2-2m-c)^2-1}-\frac{C}{(2-2m+c)^2-1}\right).$$

If we compare this with the former instance, we shall find that the introduction of the condition of a small eccentricity (so small that all terms involving its square, &c. are neglected) increases the value of u *in the first approximation*, by two new terms, corresponding to equations of which the *arguments* (see *Astron.* p. 324.) are

$$2v-2mv-cv,\ \text{and}\ 2v-2mv+cv.$$

The coefficients A, B, C, involve the disturbing force, and therefore, according to the hypothesis, are very small: if they were not, the value of u, which results from the approximate integration of the differential equation, could not agree with the assumed value, namely $\frac{1}{a}+\frac{e}{a}\cos.cv$. It can never agree exactly with it. But if we suppose them nearly equal, then, by the comparison of like terms, we shall be able to determine the value of c, which hitherto has been supposed an arbitrary quantity. And if we determine c, we shall know $1-c$, which denotes the progression of the apogee, and thence obtain (if we concede the calculation to have been rightly conducted) a test of the truth of Newton's Law of Gravitation.

According to this method, Clairaut (see *Mem. Acad.* 1745.) first proceeded and reasoned. The assumed equation

$$u = \frac{1}{a} + \frac{e}{a} \cos. cv,$$

corresponding to the assumption of the body's place in the periphery of a moveable ellipse, was nearly equal to the *resulting* value of u; that which represented, or nearly so, the inverse radius vector of the real orbit. Thence the following equations, arising from the comparison of terms, would be nearly true:

$$\frac{1}{a} = \frac{1}{a_\backslash} - \frac{K}{2a_\backslash},$$

$$\frac{e}{a} \cos. cv = -\frac{3Ke}{2a_\backslash (c^2 - 1)} \cos. cv,$$

and from the last, we have

$$c^2 = 1 - \frac{3Ka}{2a_\backslash}.$$

Now, $a_\backslash$ (see p. 133.) is what the Moon's mean distance would have been, had there been no disturbing force. But, since that force always acts, $a_\backslash$ is no real quantity, such as can be found by observation. It must be determined by calculation: and the equation of p. 133. is sufficient for that purpose; thence

$$a_\backslash = a\left(1 - \frac{K}{2}\right),$$

and we have now to compute $K \left(= \frac{m' a^3}{a'^3}\right)$. In order to compute the Moon's mean motion, we have (see p. 95.)

$$dt = \frac{dv}{hu^2 \sqrt{\left(1 + \frac{2}{h^2}\int \frac{T dv}{u^3}\right)}}.$$

Now, (see p. 131.) the constant part of u is to be represented (when the plane of the Moon's orbit is supposed to be not inclined to that of the ecliptic by $\frac{1}{a}(1-e^2)$: and since (see p. 131.) in the same case, $h=\sqrt{[a_{,}(1-e^2)]}$, we have the constant part of $\frac{1}{hu^2}$ equal $\frac{a^2}{\sqrt{a_{,}}}$, nearly: and consequently, the part of the expansion of dt not involving cosines or sines, and therefore not periodic, would be $\frac{a^2}{\sqrt{a_{,}}}.dv$:

$$\text{hence, } \frac{a^2}{\sqrt{a_{,}}}=\frac{1}{n},$$

$$\text{and, see pp. 29, \&c. } a'^{\frac{3}{2}}=\frac{\sqrt{m'}}{n'};$$

$$\therefore \frac{m'a^3}{a'^3}=\left(\frac{n'}{n}\right)^2.\frac{a_{,}}{a}=m^2\frac{a_{,}}{a},$$

$$\text{or, } K=\frac{m^2 a_{,}}{a}\left(=m^2-\frac{m^4}{2},\ \text{nearly}\right);$$

$$\therefore c^2=1-\frac{3m^2}{2}.$$

Now (see *Astronomy*, p. 308, and p. 132*.)

$$m=.0748013;\ \therefore \frac{3}{2}m^2=\frac{3}{2}(.005595)=.00838,$$

$$c^2=.9916,\ \text{nearly, and } c=.9957.$$

Hence, the progression of the apogee, whilst the Moon describes the angle v, is $(1-c)v$, which is equal to $.0042.v$, nearly, and consequently the progression in a whole revolution $=.0042\times 360^{\circ}$ $=1^{\circ}\ 30'\ 43''$, nearly, a quantity about half of that ($3^{\circ}\ 2'\ 22''$), which is determined by the most accurate observations.

This is a brief notice and description of that notorious error, which, on its first appearance, caused (if we may so express ourselves) so great a sensation in the mathematical world. In one of the most remarkable of the heavenly phenomena, the progressions of the aphelia of the planetary orbits, theory and calculation were erroneous to the amount of half the real quantity. So erroneous a defalcation seemed to portend to Newton's System, that fate which, not long before, Descartes's had experienced.

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* In p. 132. for .01748013, read .0748013.

But it may be said, that the preceding solution, with regard to the Moon, must be inexact. For, in the first place, no account is made of those terms which involve the square and higher powers of e: secondly, the plane of the Moon's orbit is supposed, (contrary to the fact) to be coincident with the plane of the equator: and thirdly, the solar orbit is supposed to be circular, whereas its eccentricity (e') is equal (see *Astron.* Chap. XVIII.) .016814: these are obvious causes of incorrectness, which, by the mere labour of calculation, may be removed: and amongst the results of that calculation it would appear, whether their removal would cause the error of the computed quantity of the apogee to disappear also.

By simply following then the natural course of successive corrections, we shall arrive at that point. It will be seen that the erroneous determination of the progression does not depend on any of the causes just enumerated. Its cause will be detected in that method of approximation, to which the imperfection of analytical science obliges us to have recourse.

By the expression, natural course of successive corrections, is meant, the addition of small quantities to terms already computed, as corrections due to those terms on account of conditions before omitted and now supplied; or, of conditions previously simplified and now more nearly restored to their true state. For instance, the solar orbit being nearly elliptical, and having been (see p. 128. 137.) supposed circular, its eccentricity is a condition omitted. To correct the error arising from this omission, we must compute several small terms for the purpose of augmenting the component parts of Π (see pp. 128. 137.). Again, the plane of the Moon's orbit, having been (see the same pages) supposed coincident with the ecliptic, its inclination is a condition omitted: and the restoring of this condition will increase Π by several small terms. The rejection, however, of all terms that involve the square and higher powers of the eccentricity (e) is to suppose one condition of the problem more simple than it ought to be assumed: for, in the Lunar Theory, $e = .0548729$; and even Clairaut, Dalembert, and Thomas Simpson who first treated of it, have extended their approximations beyond the first powers of the eccentricity.

In the next Chapter the approximation will be extended, so as to include the terms that involve e^2. An extension requiring no new principle or process, but merely a greater length of calculation.

By taking account of the terms that involve e^2, e^3, &c. we obtain, after integrating the first equation (see p. 144.), a more correct value of u. The same end is attained, when the eccentricity of the Solar Orbit and the inclination of the planes of the two orbits are introduced, as conditions of the problem. But it is not merely a more correct value of u that is obtained by the introduction of these two latter conditions. A rise is given to new equations, (see *Astronomy*, pp. 150, &c.). We can never, by calculation, account for the *annual equation* (see *Astronomy*, p. 328.) whilst u' is considered equal to $\frac{1}{a'}$; for that equation depends on the eccentricity e': nor can we, as it is plain, compute either the regression of the nodes, or the variation of the inclination of the plane of the orbit, whilst, by assuming a certain value for u, we tacitly assume the body to have no latitude.

These points, which are now only slightly glanced at, will be more fully discussed in a subsequent part of this Work.

CHAP. X.

On the Form of the Differential Equation, when the Approximation includes Terms that involve e^2. *The Error, in the Computed Quantity of the Apogee, the same as before, and very little lessened by taking account of Terms involving* e^3.

THE analytical values of P, T, remain the same as in the former case: the first alteration necessary to be made is in the values of sin. $2\,\omega$, cos. $2\,\omega$.

Values of cos. $2\,\omega$, *sin.* $2\,\omega$, *(see pp.* 129. 139.).

Since the square of e is to be retained, we shall have (see p. 139.),

$$h = \sqrt{[a_{,}(1 - e^2)]}, \;= \sqrt{a_{,}}\left(1 - \frac{e^2}{2}\right), \text{ nearly,}$$

$$u = \frac{1 + e\cos.\,c\,v}{a\,(1 - e^2)}, \text{ or } = \frac{1}{a}\,(1 + e^2 + e\,.\cos.\,c\,v),$$

$$\text{and } dt = \frac{d\,v}{hu^2} = \frac{a^2\,d\,v}{h}\,.\,(1 + e^2 + e\,.\cos.\,c\,v)^{-2}$$

$$= \frac{a^2\,.\,d\,v}{h}\,(1 - 2\,e^2 - 2\,e\cos.\,c\,v + 3\,e^2\,.\cos.^2\,c\,v)$$

rejecting the terms that involve e^3, &c.

Hence, since $\cos.^2\,c\,v = \frac{1}{2} + \frac{1}{2}\cos.\,2\,c\,v$,

$$d\,t = d\,v\,.\,\frac{a^2}{\sqrt{a_{,}}}\left(1 - 2\,e\,\cos.\,c\,v + \frac{3\,e^2}{2}\cos.\,2\,c\,v\right),$$

$$\text{making } \frac{a^2}{\sqrt{a_{,}}} = \frac{1}{n}, \text{ and, integrating,}$$

$$n\,t = v - 2\,e\sin.\,c\,v + \frac{3\,e^2}{4}\sin.\,2\,c\,v,$$

c, which ought to appear in the denominators of the second and third term, being supposed $=1$. We have therefore, instead of the former equation (see p. 140.), for deducing $\sin. 2\omega$ and $\cos. 2\omega$, this (the solar orbit being still considered circular),

$$\omega = v - v' = v(1-m) + 2em \sin. cv - \frac{3e^2 m}{4} \sin. 2cv,$$

and thence we have

$$\cos. 2\omega = \cos. [2v.(1-m) + 4em \sin. cv]$$
$$+ \frac{3e^2 m}{4} \sin. 2cv . \sin. [2v(1-m) + 4em \sin. cv],$$

the cosine of $\frac{3e^2 m}{4} \sin. 2cv$ being nearly $= 1$, and the sine of the same quantity being nearly the quantity itself (see *Trig.* p. 104.)

Now the cosine and sine of $2v.(1-m) + 4em \sin. cv$, are the cosine and sine of 2ω in the former case * (see p. 140.): consequently, since

$$\sin. 2cv . \sin. (2v - 2mv) =$$
$$\frac{1}{2}[\cos. (2v - 2mv - 2cv) - \cos. (2v - 2mv + 2cv)]$$

$$\begin{aligned}\cos. 2\omega = \cos. (2v - 2mv) &- 2me . \cos. (2v - 2mv - cv)\\ &+ 2me . \cos. (2v - 2mv + cv)\\ &+ \frac{3me^2}{4} \cos. (2v - 2mv - 2cv)\\ &- \frac{3me^2}{4} \cos. (2v - 2mv + 2cv)\end{aligned}$$

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* The $\cos. [2v . (1-m) + 4em . \sin. cv]$
$= \cos. 2v . (1-m) . \cos. 4em . \sin. cv$
$- \sin. 2v . (1-m) . \sin. 4em . \sin. cv$;

now, instead of $\cos. 4em . \sin. cv$, we have written 1 (rad.) from the smallness of $4em . \sin. cv$: but a nearer value is

$$1 - \frac{(4em . \sin. cv)^2}{2}, \text{ or } 1 - 4e^2 m^2 + 4e^2 m^2 \cos. 2cv:$$

consequently, the succeeding values of $\cos. 2\omega$, which are used in the text, depend on the rejection of terms involving $(em)^2$, which, in the Lunar Theory, is very small.

and similarly,

$$\begin{aligned}\sin.\,2\,\omega = \sin.\,(2\,v - 2\,m\,v) &- 2\,m\,e\,.\,\sin.\,(2\,v - 2\,m\,v - c\,v)\\ &+ 2\,m\,e\,.\,\sin.\,(2\,v - 2\,m\,v + c\,v)\\ &+ \frac{3\,m\,e^2}{4}\,.\,\sin.\,(2\,v - 2\,m\,v - 2\,c\,v)\\ &- \frac{3\,m\,e^2}{4}\,.\,\sin.\,(2\,v - 2\,m\,v + 2\,c\,v)^*.\end{aligned}$$

Value of $\frac{1}{h^2}\ \frac{T}{u^3}\ \frac{du}{dv}$, (*see pp.* 130. 142.).

$$u = \frac{1}{a}\,(1 + e^2 + e\,.\,\cos.\,c\,v).$$

But, since it is not intended to include terms that involve e^3, &c., we have, as before,

$$\frac{du}{dv} = -\,\frac{c\,e}{a}\,\sin.\,c\,v,$$

and $\frac{1}{h^2}\ \frac{T}{u^3}\ \frac{du}{dv} = \frac{3\,K\,c\,e}{2\,a}\,(1 - 4\,e\,.\,\cos.\,c\,v)\ \sin.\,c\,v\,.\,\sin.\,2\,\omega$;

now,

$$\begin{aligned}e\,\sin.\,c\,v\,.\,\sin.\,2\,\omega = \frac{e}{2}\,&[\cos.\,(2\,v - 2\,m\,v - c\,v) - \cos.\,(2\,v - 2\,m\,v + c\,v)]\\ - \frac{2\,m\,e^2}{2}\,&[\cos.\,(2\,v - 2\,m\,v - 2\,c\,v) - \cos.\,(2\,v - 2\,m\,v)]\\ + \frac{2\,m\,e^2}{2}\,&[\cos.\,(2\,v - 2\,m\,v) - \cos.\,(2\,v - 2\,m\,v + 2\,c\,v)],\end{aligned}$$

and

$$4\,e^2\,.\,\cos.\,c\,v\,\sin.\,c\,v\,.\,\sin.\,2\,\omega = 2\,e^2\,[\sin.\,2\,c\,v\,.\,\sin.\,(2\,v - 2\,m\,v)]$$

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[*] This method is easily extended to find cos. $2\,\omega$, sin. $2\,\omega$, when the terms that involve e^3, &c. are taken account of: for if

$$2\,\omega' = 2\,\omega + 2\,e^3\,\sin.\,2\,x,\ \text{then}$$

$$\cos.\,2\,\omega' = \cos.\,2\,\omega - e^3\,[\cos.\,(2\,\omega - 2\,x) - \cos.\,(2\,\omega + 2\,x)].$$

Again, if $2\,\omega'' = 2\,\omega' - 2\,e^4\,.\,\sin.\,2\,z$,

$$\cos.\,2\,\omega'' = \cos.\,2\,\omega' - e^4\,.\,[\cos.\,(2\,\omega' - 2\,z) - \cos.\,(2\,\omega' + 2\,z)].$$

$$= e^2\,[\cos.\,(2v - 2mv - 2cv) - \cos.\,(2v - 2mv + 2cv)].$$

Hence,

$$\frac{1}{h^2}\,\frac{T}{u^3}\,\frac{du}{dv} = \frac{3Kce}{4a}\left\{\begin{array}{l}\cos.\,(2v - 2mv - cv)\\ -\cos.\,(2v - 2mv + cv)\\ -2e\,.\,(1+m)\cos.\,(2v - 2mv - 2cv)\\ +2e\,.\,(1-m)\,.\cos.\,(2v - 2mv + 2cv)\\ +4me\,.\cos.\,(2v - 2mv)\end{array}\right\}$$

Here we see that the extending the approximation, so as to include those terms that involve e^2, adds to the value of the preceding quantity three new terms, and consequently must add three new *equations* (see *Astronomy*, pp. 324, &c.) for determining and correcting the value of u. The *arguments* of the new equations (one of which has occurred before) are

$2v - 2mv - 2cv$, $2v - 2mv + 2cv$, and $2v - 2mv$;

Value of $\frac{\mathrm{P}}{\mathrm{h}^2\,\mathrm{u}^2}$ (*see pp.* 129, 138, 143.)

$$\frac{P}{h^2 u^2} = \frac{1}{h^2}\left(1 - \frac{m' u'^3}{2u^3} - \frac{3m'}{2}\,\frac{u'^3}{u^3}\cos.\,2\,\omega\right)$$

$$\frac{m' u'^3}{2h^2 u^3} = \frac{K}{2h^2}\,(1 + e^2 + e\,.\cos.\,cv)^{-3}$$

$$* = \frac{K}{2h^2}\,(1 - 3e^2 - 3e\cos.\,cv + 6e^2\,.\cos.^2 cv)$$

$$= \frac{K}{2a}\,(1 + e^2 - 3e\cos.\,cv + 3e^2\cos.\,2cv).$$

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* In deducing the progression of the Lunar Apogee, it is necessary to compute to the greatest exactness the coefficient of cos. cv: now in the expansion of $(1 + e\cos.\,cv)^{-3}$, the term involving the cube of cos. cv is $-\,10e^3\cos.^3 cv = -\,10e^3\left(\frac{\cos.\,3cv}{4} + \frac{3}{4}\cos.\,cv\right)$; hence, the whole coefficient of cos. cv will be $-\,3e - \frac{3\,.\,5}{2}e^3$, and consequently in $\frac{(1+e^2)^{-3}}{h^2}\,(1 + e\,.\cos.\,cv)^{-3}$ will be $-\,3e + \frac{e^3}{2}$.

Secondly,

$$\frac{3\,m'\,u'^3}{2\,h^2\,u_{,}^3}\cos.\,2\,\omega =$$

$$\frac{3\,K}{2\,h^2}\,(1 - 3\,e\cos.\,c\,v + 3\,e^2\cos.\,2\,c\,v)\times$$

$$\left\{\begin{array}{l}\cos.\,(2v-2mv) - 2me\,.\cos.\,(2v-2mv-cv) + 2me\cos.(2v-2mv+cv)\\ \quad + \dfrac{3\,m\,e^2}{4}\cos.\,(2\,v-2m\,v-2\,c\,v) - \dfrac{3me^2}{4}\cos.\,(2v-2\,m\,v+2\,cv)\end{array}\right\}$$

= (when the terms are combined and expanded, and the *coefficients* of like arguments collected together),

$$\frac{3\,K}{2\,a_{,}}\left\{\begin{array}{l}(1 + e^2)\,.\cos.\,(2\,v - 2\,m\,v)\\ -\dfrac{3+4m}{2}\,e\,.\cos.\,(2\,v - 2\,m\,v - c\,v)\\ -\dfrac{3-4m}{2}\,e\,.\cos.\,(2\,v - 2\,m\,v + c\,v)\\ +\dfrac{6+15m}{4}\,e^2\,.\cos.\,(2\,v - 2\,m\,v - 2\,c\,v)\\ +\dfrac{6-15m}{4}\,e^2\,.\cos.\,(2\,v - 2\,m\,v + 2\,c\,v).\end{array}\right\}$$

Since in the Lunar Theory ($c + m > 1$ $c = .991548$ $m = .0748013$), the latter cosines may be thus written :

$$\cos.\,(2\,c\,v + 2\,m\,v - 2\,v),\ \cos.\,(2\,c\,v - 2\,m\,v + 2\,v).$$

Value of $\frac{2}{\mathrm{h}^2}\int\frac{\mathrm{T\,d\,v}}{\mathrm{u}^3}$ (*see pp.* 130, 143.)

$$\frac{T\,d\,v}{u^3} = -\frac{3\,m'}{2}\,\frac{u'^3}{u^4}\sin.\,2\,\omega\,.\,d\,v =$$

$$-\frac{3\,K\,a}{2}\,d\,v\,(1 + e^2 - 4\,e\,.\cos.\,c\,v + 5\,e^2\cos.\,2\,c\,v)\times$$

$$\left\{\begin{array}{l}\sin.\,(2v-2mv) - 2me\sin.\,(2v-2mv-cv) + 2me\sin.\,(2v-2mv+cv)\\ \quad + \dfrac{3\,m\,e^2}{4}\,.\sin.\,(2\,v-2\,mv-2\,cv) - \dfrac{3me^2}{4}\sin.\,(2v-2mv+2cv)\end{array}\right\}$$

$$= -\frac{3Ka}{2}\,dv \left\{ \begin{array}{l} (1+e^2)\,.\,\text{sin}\,.(2v-2mv) \\ -\,2\,.\,(1+m)e\,.\,\text{sin.}\,(2v-2mv-cv) \\ -\,2\,.\,(1-m)e\,.\,\text{sin.}\,(2v-2mv+cv) \\ -\,\dfrac{10+19m}{4}\,e^2.\,\text{sin.}\,(2cv-2v+2mv) \\ -\,\dfrac{10-19m}{4}\,e^2\,.\,\text{sin.}\,(2cv+2v-2mv) \end{array} \right\}$$

now this quantity must be divided by h^2, or multiplied by $\frac{1}{a_{,}\,(1-e^2)} = \frac{1}{a_{,}}\,.\,(1+e^2)$, nearly; but such multiplication, since the terms involving e^3, &c. are to be excluded, will affect only the coefficient of the first term (within the brackets), and that it will make $1+2e^2$. If, after this operation, the integral be taken,

$$\frac{2}{h^2}\int\frac{T\,dv}{u^3} = \frac{3Ka}{a_{,}} \left\{ \begin{array}{l} \dfrac{1+2e^2}{2-2m}\,.\,\text{cos.}\,(2v-2mv) \\ -\,\dfrac{2\,.\,(1+m)}{2-2m-c}\,e\,.\,\text{cos.}\,(2v-2mv-cv) \\ -\,\dfrac{2\,.\,(1-m)}{2-2m+c}\,e\,.\,\text{cos.}\,(2v-2mv+cv) \\ -\,\dfrac{10+19m}{4.(2c-2+2m)}\,e^2.\,\text{cos.}\,(2cv-2v+2mv) \\ +\,\dfrac{10-19m}{4.(2c+2-2m)}\,e^2\,.\,\text{cos.}\,(2cv+2v-2mv) \end{array} \right\}$$

Value of $\frac{d^2u}{dv^2}+u$ (*see p.* 144.)

$$u = \frac{1}{a}\,(1+e^2+e\,\text{cos.}\,cv);$$

$$\therefore\ \frac{d^2u}{dv^2}+u = \frac{1}{a}\,(1+e^2) + \frac{e}{a}\,(1-e^2)\,.\,\text{cos.}\,cv.$$

Hence,

$$\left(\frac{d^2u}{dv^2}+u\right)\frac{2}{h^2}\int\frac{T\,dv}{u^3}$$

$$=\frac{3K}{a_{,}}\left\{\begin{array}{l}\dfrac{1+3e^2}{2-2m}\,.\cos.\,(2v-2mv)\\[2ex] +\dfrac{1-c^2}{4(1-m)}-\dfrac{2.(1+m)}{2-2m-c}\,e\,.\cos.\,(2v-2mv-cv)\\[2ex] +\dfrac{1-c^2}{4.(1-m)}-\dfrac{2.(1-m)}{2-2m+c}\,e\,.\cos.(2v-2mv+cv)\\[2ex] -\dfrac{10+19m}{4\,.\,(2c-2+2m)}\,e^2\,.\cos.\,(2cv-2v+2mv)\\[2ex] +\dfrac{10-19m}{4\,.\,(2c+2-2m)}\,e^2\,.\cos.\,(2cv+2v-2mv)\end{array}\right\}$$

If we now collect (as before p. 144.) and arrange the terms of the differential equation, we shall have

$$0=\frac{d^2u}{dv^2}+u-\frac{1}{a_{,}}(1+e^2)+\frac{K}{2a_{,}}(1+e^2)-\frac{3\,K}{2a_{,}}\,e\,.\cos.\,cv$$

$$+\frac{3Ke^2}{2a_{,}}\,.\cos.\,2cv+\frac{3K}{2a_{,}}\left(1+(1+2mc)e^2+\frac{1+3e^2}{1-m}\right)\cos.\,(2v-2mv)$$

$$+\frac{3K}{a_{,}}\left(\frac{c}{4}-\frac{3+4m}{4}-\frac{2.(1+m)}{2-2m-c}+\frac{1-c^2}{4.(1-m)}\right)e\,.\cos.\,(2v-2mv-cv)$$

$$-\frac{3K}{4a_{,}}\left(c+3-4m+\frac{8\,.(1-m)}{2-2m+c}+\frac{1-c^2}{1-m}\right)e\,.\cos.\,(2v-2mv+cv)$$

$$+\frac{3K}{4a_{,}}\left({}^{*}\,\frac{2+11m}{2}-\frac{10+19m}{2c-2+2m}\right)e^2.\cos.\,(2cv-2v+2mv)+\&c.$$

If we integrate this equation by the method of p. 100. there

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[*] For

$$-2ce^2(1+m)+\frac{6+15m}{2}e^2=\frac{2+11m}{2},\ \text{nearly,}$$

when c nearly $=1$.

will result a value of u more correct than that which was there obtained.

But the error in the progression of the apogee determined by the comparison of the coefficients of cos. $c\,v$ in the assumed and resulting values of u (see pp. 146, &c.) remains unaltered: and it is easy to see, if the approximation were extended so as to include terms involving e^3, that the term involving cos. $c\,v$ in the differential equation would be merely

$$-\frac{3\,K}{2\,a_{\prime}}\left(1+\frac{e^2}{2}\right)e\,.\,\text{cos.}\;c\,v,$$

and the corresponding term in the integral equation

$$-\frac{3\,K}{2\,a_{\prime}\,(c^2-1)}\left(1+\frac{e^2}{2}\right)e\,.\,\text{cos.}\;c\,v,$$

which, equated with $\frac{1}{a}(1+e^2)\,e\,.\,\text{cos.}\;c\,v$ in the assumed value of u (see p. 146.), gives

$$c^2-1=-\frac{3\,K\,a}{2\,a_{\prime}}\left(1-\frac{e^2}{2}\right),$$

$$\text{and } c=\sqrt{\left[1-\frac{3\,K}{2}\;\frac{a}{a_{\prime}}\left(1-\frac{e^2}{2}\right)\right]};$$

which is a value a little more near to the true value than the one given in p. 146, but still very distant from it.

The introduction then, of terms involving e^2, e^3, &c. produces but little change in the resulting value of c: and hence we may presume that its just correction would not ensue, by taking account of two other omitted conditions, namely, the inclination of the plane of the Moon's orbit, and the eccentricity of the Solar. We must search elsewhere for the source of error.

But, at present, the determination of the exact quantity of the progression of the Lunar Apogee is a collateral object. The series of corrections begun will be continued for the purpose of obtaining a more exact value of u; and the next step will be to introduce the conditions of the eccentricity of the Solar orbit and of the inclination: the error arising from their previous omission will then be corrected by a method which will serve to correct the process of approximation itself. It is this last kind of correction which is, above all others, the most important.

CHAP. XI.

On the Corrections due to the Eccentricity of the Solar Orbit, and to the Inclination of the Plane of the Moon's Orbit. Method of deriving Corrections. Their Formulæ exhibited in a Table. The Error in the determination of the Lunar Apogee not removed by these Corrections. The deduction of Terms on which the Secular Equations of the Moon's Mean Longitude and of the Progression of the Apogee depend.

THE immediate object of the introduction of the conditions of the inclination and the eccentricity of the Solar orbit, is, to obtain a more correct value of Π in the differential equation

$$\frac{d^2 u}{d v^2} + u + \Pi = 0,$$

and thence, by integration, a more exact value of u.

It will easily be perceived, that the suppression of the abovementioned conditions must render the expression for Π inaccurate. For, consider one of its component parts $\frac{3\, m'\, u'^3}{2\, h^2\, u^3} \,.\, \cos.\, 2\,\omega$.

First, $\frac{1}{h^2}$ instead of being $= a_{,}\,(1 - e^2)$ is $= a_{,}\,(1 - e^2 - \gamma^2)$.

Secondly, in u'^3, u' was made $= \frac{1}{a'}$, whereas, if we suppose the Solar orbit to be like the Lunar, nearly an ellipse with a progressive apogee ($1 - c'$ denoting the progression), we ought to assume for u' an equation similar to the one for u (see pp. 140. 150.), and to make

$$u' = \frac{1}{a'}(1 + e'^2 + e' \cos. c'v) = \frac{1}{a'}(1 + e' \cos. c'v)$$

rejecting the terms that involve e'^2, &c.

Thirdly, in $\frac{1}{u^3}$, u was made $= \frac{1}{a}(1 + e^2 + e . \cos. cv)$, whereas, if the orbit be inclined, and γ be the tangent of its inclination,

$$u = \frac{1}{h^2(1+\gamma^2)}\left(1 + e \cos. cv + \frac{\gamma^2}{4} - \frac{\gamma^2}{4}\cos. 2gv\right) *.$$

Lastly, the value of cos. $2\,\omega$ was obtained (see pp. 139. 150.) by equating the two expressions for the time which are both inaccurate, since they were computed from

$$dt = \frac{dv}{hu^2},\ \text{and}\ dt = \frac{dv'}{h'u'^2},$$

after that *defective* values had been substituted for u and u': on both accounts then cos. $2\,\omega$ must be an inexact value, or will require two corrections.

Corrected Values of sin. $2\,\omega$, *cos.* $2\,\omega$ *(see pp.* 139. 150.).

The value of u' being precisely similar to that of u (see

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[*] In the note to p. 38. the last term of the value of u is $-\frac{\gamma^2}{4}\cos. 2v$, which is right when there is no disturbing force: but as (see pp. 110, &c.) the first approximate value of u obtained from the differential equation

$$\frac{d^2u}{dv^2} + u + \Pi = 0$$

involves not cos. v, but cos. cv, so the first approximate value of s to be obtained by the integration of

$$\frac{d^2s}{dv^2} + s + \Sigma = 0,$$

an equation similar to the former, will involve not sin. v, but sin. gv: and as, in the assumption of a value of u, the object is to assume one as near to the true value as we can, so the one in the text is assumed instead of that of p. 38. which belongs to the elliptical and undisturbed system.

p. 150.) there must result a similar equation for the time: accordingly,

$$n't = v' - \frac{2e'}{c'} \sin. c'v' \doteq v' - 2e' \sin. c'v',$$

since the denominator c' may be made $= 1$. By equating now the two expressions for the time, we shall have, instead of the former expression for v', (see p. 151.) this

$$v' = mv - 2me \sin. cv + \frac{3me^2}{4} \sin. 2cv + 2e' \sin. c'v',$$

in which, however, the value of v' is partly expressed by a function of v', namely, $2e' \sin. c'v'$: in order therefore to get rid of this term, multiply both sides of the preceding equation by c', and then by processes similar to those used in pp. 151, &c.* find by approximation, $\sin. c'v'$: its value will be

$$\begin{aligned} \sin. c'v' \doteq \sin. c'mv &- mec' \sin. (cv + c'mv) \\ &- mec' \sin. (cv - c'mv) \\ &+ \&c. \end{aligned}$$

If we now restore this value to the right hand side of the preceding equation, and (since c' is nearly $= 1$) make $mec' = me$, we shall have

$$\begin{aligned} v' = mv - 2me \sin. cv &+ \frac{3m}{4} . e^2 \sin. 2cv \\ &+ 2e' . \sin. c'mv \\ - 2mee' [\sin. (cv + c'mv) &+ \sin. (cv - c'mv)] \\ + \&c. & \end{aligned}$$

In this expression, the terms, after those in the first line, are an addition or increment to the value of v' arising from $\frac{e'}{a'} \cos. c'mv$ the increment of u'; or, if we wish to employ (which it is convenient to do) the symbol (δ) of *variations*, the incremental terms may be considered as variations ($\delta v'$) of v' arising from the variation ($\delta v'$) of u'.

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[*] See *Trigonometry*, p. 103.

But if v' be said to have a variation arising from $\delta u'$, it must also have one arising from u; for, the introduction of the inclination of the planes will add some small incremental terms to u, some therefore (see p. 150.) to the value of t, and accordingly, after equating the two values of t, some to the value of v'. Now

$$u = \frac{1}{h^2(1+\gamma^2)}\left(1 + e \, . \cos. cv + \frac{\gamma^2}{4} - \frac{\gamma^2}{4} \cos. 2gv\right);$$

therefore

$$\frac{1}{u^2} = h^4 \left\{ \begin{aligned} &1 + \frac{3e^2}{2} + \frac{3\gamma^2}{2} \\ &-2e\left(1 + \frac{3e^2}{2} + \frac{5}{4}\gamma^2\right) \cos. cv \, {}^{*} \\ &+ \frac{\gamma^2}{2} \cos. 2gv + \frac{3e^2}{2} \cos. 2cv \\ &- \frac{3}{4} e \gamma^2 \left[\cos. (2gv + cv) + \cos. (2gv - cv)\right] \end{aligned} \right.$$

If this expression be substituted in $dt = \dfrac{dv}{hu^2}$,

$$h^3\left(1 + \frac{3e^2}{2} + \frac{3\gamma^2}{2}\right) \text{ made } = a^{\frac{3}{2}} = \frac{1}{n},$$

the integral taken and the denominators $4g$, $2g - c$ made what they are equal to, 4 and 1 respectively, there will result very nearly †,

$$\begin{aligned} nt = v - 2e \sin. cv &+ \frac{3}{4} e^2 \cdot \sin. 2cv \\ &+ \frac{\gamma^2}{4} \sin. 2gv - \frac{3}{4} \sin. (2gv - cv). \end{aligned}$$

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* 'In deducing the coefficient of cos. cv the expansion is continued till $e^3 \cos.^3 cv$ is included, which (see *Trig.* ed. 2. p. 53.)

$$= \tfrac{3}{4} \cos. cv + \tfrac{1}{4} \cos 3cv.$$

† The coefficient of cos. $cv = -2e\left(1 + \dfrac{3e^2}{2} + \dfrac{5\gamma^2}{4}\right)$, and this divided by $1 + \dfrac{3e^2}{2} + \dfrac{3\gamma^2}{2}$ gives $-2e\left(1 - \dfrac{\gamma^2}{4}\right)$, which, since $c\gamma^2$ is extremely small, is nearly $= -2e$. And the other expressions are, in like manner, rendered more simple.

In like manner, if we were to increase the value of u, by any other additional term such as $\frac{Q}{h^2}\cos. q v$, the value of $n t$ would (Q being very small) be increased, very nearly, by

$$\frac{3Q}{2} - \frac{2Q}{q}\sin. qv + \frac{3Q}{q-c}e.\sin.(qv - cv) + \frac{3Q}{q+c}e\sin.(qv + cv).$$

If, with these additional terms, we now equate the two values of t (see p. 140.) there will result

$$v' = mv - 2me\sin. cv + \frac{3}{4}me^2\sin. 2cv$$

$$+ \frac{1}{4}m\gamma^2\sin. 2gv - \frac{3}{4}me\gamma^2\sin.(2gv - cv)$$

$$+ 2e'\sin. c'mv - 2mee'\left[\sin.(cv + c'mv) + \sin.(cv - c'mv)\right]$$

$$- \frac{2Q}{q}m\sin. qv + \frac{3Q}{q-c}me.\sin.(qv - cv) + \&c.$$

The first line in the preceding value of v', is its value (see p. 151.) when the inclination and the eccentricity of the Solar Orbit were supposed nothing. The second line is the increment of v', when the condition of the inclination of the plane of the Lunar Orbit is restored, or when

$$\delta u = \frac{1}{a}\left(\frac{\gamma^2}{4} - \frac{\gamma^2}{4}\cos. 2gv\right).$$

The third line consists of terms that are incremental to v', when $a'e'$ is supposed to be the eccentricity of the Sun's *elliptical* orbit, and when

$$\delta u' = \frac{e'}{a}, \cos. c'mv.$$

The fourth line is the increase to the value of v', when we either supply a *deficient* * term to the value of u, or increase that value by an *acquired* term: and when, according to the notation that has been adopted,

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* u, by the successive integrations of the differential equation, acquires new terms: these cannot be known till after integration: the *deficient* terms of u are those which, for the sake of simplicity, are omitted in the assumed value of u.

$$\delta u = \frac{Q}{h^2} \cos.\, q\, v, \text{ or, nearly, } = \frac{Q}{a} \cos.\, q\, v.$$

We may now easily deduce $\delta\,(\sin.\, 2\,\omega)$, and $\delta\,(\cos.\, 2\,\omega)$, or those variations of sin. 2ω and cos. $2\,\omega$, which are respectively due to the preceding variations in the values of u and u'. Thus

$$\delta\,.\,\sin.\, 2\omega = \delta\,.\,\sin.\,(2v - 2\,v') = -\,2\,\delta\, v'\,.\,\cos.\, 2\,\omega,$$
$$\delta\,.\,\cos.\, 2\,\omega = \delta\,.\,\cos.\,(2\,v - 2\,v') = \quad 2\,\delta\, v'\,.\,\sin.\, 2\,\omega.$$

If therefore the variation of sin. 2ω, arising from $\delta u' = \frac{e'}{a'} \cos.\, c' m v$, be required, we have only to substitute, instead of $d\,v'$, the third line of the preceding value of v'; and still more simple will be the process, if the terms involving $e\,e'$ are to be excluded by reason of their smallness: for then it will be sufficient to make $\delta\, v' = 2\, e'\, \sin.\, c'\, m\, v$; and in this case (see *Trig.* p. 26. [*b*]),

$$\delta\,.\,\sin.\, 2\,\omega = 2\, e'\,.\,\sin.\,(2\,v - 2\,m\,v - c'\,m\,v)$$
$$-\,2\,e'\,.\,\sin.\,(2\,v - 2\,m\,v + c'\,m\,v).$$

Similarly, $\delta\,.\,\cos.\, 2\,\omega = 2\,e'\,.\,\cos.\,(2\,v - 2\,m\,v - c'\,m\,v)$
$$-\,2\,e'\,.\,\cos.\,(2\,v - 2\,m\,v + c'\,m\,v).$$

If it is necessary to express the terms that involve $e\,e'$, then, instead of $\delta\, v'$, we must substitute the terms of the third line of the value of v', and we must also, in the process, take account of the three first terms of sin. $2\,\omega$, and cos. $2\,\omega$, as given in pp. 151, &c. there will then result by the common trigonometrical formulæ, (see *Trig.* pp. 24, &c.).

$$\delta\,.\,\sin.\, 2\,\omega = \left\{\begin{array}{l}
2\,e'\,.\,\sin.\,(2\,v - 2\,m\,v - c'\,m\,v) \\
-2\,e'\,.\,\sin.\,(2\,v - 2\,m\,v + c'\,m\,v) \\
-2\,m\,e\,e'\,.\,\sin.\,(2\,v - 2\,m\,v + c\,v + c'\,m\,v) \\
+2\,m\,e\,e'\,.\,\sin.\,(2\,v - 2\,m\,v - c\,v + c'\,m\,v) \\
+6\,m\,e\,e'\,.\,\sin.\,(2\,v - 2\,m\,v + c\,v - c'\,m\,v) \\
-6\,m\,e\,e'\,.\,\sin.\,(2\,v - 2\,m\,v - c\,v - c'\,m\,v)
\end{array}\right.$$

and a similar expression may easily be obtained for $\delta\,.\,\cos.\, 2\,\omega$.

If we wish to deduce $\delta\,.\,\sin.\, 2\,\omega$ and $\delta\,.\,\cos.\, 2\,\omega$ arising from the variation $\delta u = -\frac{\gamma^2}{4a}\cos. 2gv$, we have merely to substitute, instead of $\delta v'$, the first term $\left(\frac{m\gamma^2}{4}\sin. 2gv\right)$ of the second line of the value for v'; or, we may immediately obtain the variation, by writing, in the expression, $\frac{m\gamma^2}{4}$ instead of $2e$, and $2gv$ instead of $c'mv$: there will then result

$$\delta . \sin. 2\omega = \frac{m\gamma^2}{4}\sin. (2gv - 2v + 2mv)$$
$$- \frac{m\gamma^2}{4}\sin. (2gv + 2v - 2mv)$$

$\left(\delta u = -\frac{\gamma^2}{4a}\cos. 2gv\right)$

$$\delta . \cos. 2\omega = \frac{m\gamma^2}{4}\cos. (2gv - 2v + 2mv)$$
$$- \frac{m\gamma^2}{4}\cos. (2gv + 2v - 2mv).$$

And, after the manner of deducing the terms that involve ee', may be deduced the terms that involve $\gamma^2 e$.

In like manner,

$$\delta . \sin. 2\omega = \frac{2mQ}{q}\sin. (qv - 2v + 2mv)$$
$$+ \frac{2mQ}{q}\sin. (qv - 2v - 2mv)$$

$\left(\delta u = \frac{Q}{2}\cos. qv\right)$,

$$\delta . \cos. 2\omega = -\frac{2mQ}{q}\cos. (qv - 2v + 2mv)$$
$$+ \frac{2mQ}{q}\cos. (qv + 2v - 2mv).$$

If we now, for the purpose of exhibiting at one view, sin. 2ω and cos. 2ω, complete their values, by adding, that what have been already (see pp. 151, 164.) exhibited, as so many corrections, the terms arising from the variation $\delta u'$ and from the two variations of u, we shall have

$$
\begin{array}{ll}
\text{sin. } 2\omega = \text{sin. } (2v - 2mv) & -2me\,.\text{sin. } (2v - 2mv - cv) \\
 & +2me\,.\text{sin. } (2v - 2mv + cv) \\
 & -\dfrac{3me^2}{4}\,.\text{sin. } (2cv + 2v - 2mv) \\
 & -\dfrac{3me^2}{4}\,.\text{sin. } (2cv - 2v + 2mv) \\
\left[\delta u' = \dfrac{e'}{a'}\text{cos. } c'mv\right] & +2e'\,.\text{sin. } (2v - 2mv - c'mv) \\
 & -2e'\,.\text{sin. } (2v - 2mv + c'mv) \\
\left[\delta u = -\dfrac{\gamma^2}{4a}\text{ cos. } 2gv\right] & -\dfrac{m\gamma^2}{4}\,.\text{sin. } (2gv + 2v - 2mv) \\
 & -\dfrac{m\gamma^2}{4}\,.\text{sin. } (2gv - 2v + 2mv) \\
\left[\delta u = \dfrac{Q}{a}\text{cos. } qv\right] & +\dfrac{2mQ}{q}\,.\text{sin. } (qv + 2v - 2mv) \\
 & +\dfrac{2mQ}{q}\,.\text{sin. } (qv - 2v + 2mv) \\
 & +\ \&\text{c.}
\end{array}
$$

$$
\begin{array}{ll}
\text{cos. } 2\omega = \text{cos. } (2v - 2mv) & -2me\text{ cos. } (2v - 2mv - cv) \\
 & +2me\text{ cos. } (2v - 2mv + cv) \\
 & -\dfrac{3}{4}me^2\text{ cos. } (2cv + 2v - 2mv) \\
 & +\dfrac{3}{4}me^2\,.\text{cos. } (2cv - 2v + 2mv) \\
\left[\delta u' = \dfrac{e'}{a'}\text{cos. } c'mv\right] & +2e'\,.\text{cos. } (2v - 2mv - c'mv) \\
 & -2e'\,.\text{cos. } (2v - 2mv + c'mv) \\
\left[\delta u = -\dfrac{\gamma^2}{4a}\text{ cos. } 2gv\right] & -\dfrac{m\gamma^2}{4}\text{ cos. } (2gv + 2v - 2mv) \\
 & +\dfrac{m\gamma^2}{4}\text{ cos. } (2gv - 2v + 2mv) \\
\left[\delta u = \dfrac{Q}{a}\text{ cos. } qv\right] & +\dfrac{2mQ}{q}\text{ cos. } (qv + 2v - 2mv) \\
 & -\dfrac{2mQ}{q}\text{ cos. } (qv - 2v + 2mv). \\
 & +\ \&\text{c.}
\end{array}
$$

It is easy, after the manner of p. 164. to increase the preceding values by terms that involve $e\,e'$, &c.

We will now proceed to deduce the corrections that are due to the terms composing Π, and that arise from the eccentricity $(a'\,e')$ of the Solar orbit; excluding, however, from the formulæ of corrections, those terms that involve the square of e' or any rectangle such as $e\,e'$, $e'\,\gamma^2$, &c.

$$\frac{1}{h^4} \cdot \frac{T}{u^3}\,\frac{d\,u}{d\,v} \quad \text{(see pp. 130, 142, 152.)}$$

$\frac{d\,u}{d\,v}$ involves e, the simplest term therefore of the correction of the above term must involve $e\,e'$: of such terms, however, it is not, at the present, intended to take account,

$$\frac{m'}{2\,h^2} \cdot \frac{u'^3}{u^3} \quad \text{(see p. 153.)}$$

$$\delta\,.\,u'^3 = 3\,u'^2\,.\,\delta\,u' = \frac{3}{a'^2} \times \frac{e'}{a'}\,.\,\cos.\,c'\,m\,v.$$

Therefore the correction of $\frac{m'}{2\,h^2}\,\frac{u'^3}{u^3} = \frac{3\,K}{2\,h^2}\,e'\,.\,\cos.\,c'\,m\,v,$

$$\frac{3\,m'\,u'^3}{2\,h^2\,u^3}\cos.\,2\,\omega \quad \text{(see p. 154.)}$$

$$\delta\,(u'^3\,.\,\cos.\,2\,\omega) = 3\,u'^2\,.\,\delta\,u'\,.\,\cos.\,2\,\omega + u'^3\,.\,\delta\,(\cos.\,2\,\omega)$$

$$= \frac{3}{a'^3}\,e'\,.\,\cos.\,c'\,m\,v \times \cos.\,(2\,v - 2\,m\,v)$$

$$+ \frac{2\,e'}{a'^3}\cos.\,(2\,v - 2\,m\,v - c'mv) - \frac{2\,e'}{a'^3}\cos.\,(2\,v - 2mv + c'\,m\,v),$$

where $\cos.\,(2\,v - 2\,m\,v)$, its first term, is put for $\cos.\,2\,\omega$, since terms involving $e\,e'$ are to be excluded. The last line is composed of the terms in the fifth and sixth lines of $\cos.\,2\,\omega$ (see p. 166.).

Hence, if we expand (see *Trig.* p. 26. [*d*]) the first line of the preceding expression, and then combine the resulting terms with the terms in the last line, we shall have the required correction, or

$$\frac{3m'}{2h^2}\cdot\delta\left(\frac{u'^3}{u^3}\cos.\,2\,\omega\right) = \frac{3\,K}{2\,h^2}e'\left\{\begin{matrix}\frac{7}{2}\cos.\,(2\,v-2\,mv-c'\,m\,v)\\ -\frac{1}{2}\cos.\,(2v-2mv+c'mv)\end{matrix}\right\}.$$

$$\frac{2}{h^2}\int\frac{T\,d\,v}{u^3},\text{ (see pp. 130, 143, 154.)},$$

$$T = -\frac{3\,m'\,u'^3}{u}\sin.\,2\,\omega,\ \text{ and}$$

$$\delta\,(u'^3\sin.\,2\,\omega) = 3\,u'^2\,.\,\delta u'\,.\sin.\,2\,\omega + u'^3\,.\,\delta\,.\sin.\,2\,\omega$$

$$= \frac{3}{a'^3}\,e'\,.\cos.\,c'\,m\,v\times\sin.\,(2\,v-2\,m\,v)$$

$$+\frac{2\,e'}{a'^3}\,[\sin.\,(2\,v-2\,m\,v-c'\,m\,v)-\sin.\,(2\,v-2\,mv+c'\,m\,v)]$$

$$= \frac{e'}{a'^3}\left(\frac{7}{2}\sin.(2v-2\,mv-c'm\,v)-\frac{1}{2}\sin.\,2v-2mv+c'mv\right).$$

Hence

$$\frac{2}{h^2}\int\frac{T\,d\,v}{u^3} = \frac{3Ka}{h^2}e'\left\{\begin{matrix}\frac{7}{2.(2-2m-c'm)}\,.\cos.\,(2v-2mv-c'mv)\\ -\frac{1}{2.(2-2m+c'm)}\,.\cos.\,(2v-2mv+c'mv)\end{matrix}\right\}.$$

Since c' is very nearly $=1$ (it $=.9999907779$) the denominators in the preceding expression are nearly $2\,.\,(2-m)$, and $2\,(3-2m)$: and, in all the preceding cases, since no account is to be made of terms that involve $e^2\,e'$, $\gamma^2\,e$, &c. $\frac{1}{a}$ may be written instead of $\frac{1}{h^2}$: and for the same reason, we shall have the correction due to

$$\left(\frac{d^2u}{dv^2}+u\right)\frac{2}{h^2}\int\frac{T\,d\,v}{u^3},$$

by simply dividing the preceding expression by a.

Hence the whole correction due to Π, on account of the eccentricity of the Solar orbit and confined to terms that involve merely e', is of the form

$$Ae'.\cos.\,c'mv+Be'.\cos.\,(2v-2mv-c'mv)+Ce'.\cos.\,(2v-2mv+c'mv),$$

and of the same form (see p. 100, &c.) is the correction due to u and obtained by integration.

If we refer to page 164, we may thence easily infer that the *arguments* of the terms involving $e\,e'$ will be

$$2v - 2mv - cv - c'mv,$$
$$2v - 2mv - cv + c'mv,$$
$$2v - 2mv + cv + c'mv,$$
$$2v - 2mv + cv - c'mv.$$

Precisely after the preceding manner we may correct, the terms composing Π, on account of the plane's inclination. But, when this latter condition is introduced, Π will be represented by more terms than in pages 138. 144; for in this case, see p. 65.

$$P = u^2 (1 + s^2)^{-\frac{3}{2}} - \frac{m' u'^3}{2u} - \frac{3\,m' u'^3}{2u} \cos.\, 2\,\omega,$$

$$= u^2 - \frac{3}{2} u^2 s^2 - \frac{m' u'^3}{2u} - \frac{3\,m' u'^3}{2u} \cos.\, 2\,\omega, \text{ nearly,}$$

$$= u^2 - \frac{3}{4} u^2 \gamma^2 + \frac{3}{4} u^2 \gamma^2 \cos.\, 2\,g\,v - \frac{3\,m' u'^3}{2u} \cos.\, 2\,\omega.$$

Hence, see p. 95.

$$\Pi = \frac{1}{h^2} \cdot \frac{T}{u^3} \cdot \frac{du}{dv} - \frac{1}{h^2} + \frac{3\,\gamma^2}{4\,h^2} - \frac{3\gamma^2}{4h^2} \cos.\, 2\,g\,v$$

$$+ \frac{m' u'^3}{2 \cdot h^2 u^3} + \frac{3\,m' u'^3}{2\,h^2 u^3} \cos.\, 2\omega + \left(\frac{d^2 u}{d v^2} + u\right) \frac{2}{h^2} \int \frac{T\,dv}{u^3},$$

in which h^2, instead of its former value, equals

$$a_{,} (1 - e^2 - \gamma^2).$$

We will now proceed to deduce the several corrections; but, instead of supposing $\delta u = \frac{\gamma^2}{4} - \frac{\gamma^2}{4} \cos.\, 2\,g\,v$ (which is the increment of the *elliptical* value of u when the condition of the inclination is introduced), we will suppose

$$\delta u = \frac{Q}{a} \cos.\, q\,v,$$

and thence obtain, by substitution, the correction we are in quest of; whether it is due to a *deficient* or to an *acquired* term in the value of u.

$$\frac{1}{h^2} \cdot \frac{T}{u^3} \cdot \frac{du}{dv} \quad (\textit{see pp.}\ 130.\ 142.\ 152.\ 167.)$$

$T = -\frac{3\,m'\,u'^3}{u}\,\sin.\,2\,\omega$; now, $\frac{Q}{a}\cos.\,q\,v$ being the new term in u, $-\frac{Q\,q}{a}\sin.\,q\,v$ will be the corresponding *new* term in $\frac{du}{dv}$; and, technically,

$$\delta\left(\frac{d\,u}{d\,v}\right) = -\frac{Q\,q}{a}\,\sin.\,q\,v.$$

Now, by the rules for finding either the differentials or variations * of products,

$$\delta.\left(\frac{\sin.\,2\,\omega}{u^4}\cdot\frac{d\,u}{d\,v}\right) = \frac{1}{u^4}\cdot\frac{d\,u}{d\,v}\ \delta.\sin.\,2\,\omega$$

$$-\frac{4\,\sin.\,2\,\omega}{u^5}\ \frac{d\,u}{d\,v}.\,\delta\,u + \frac{\sin.\,2\,\omega}{u^4}\ \delta.\frac{d\,u}{d\,v}.$$

the two first terms will involve $Q\,e$, which, since it is, by hypothesis, a very small quantity, is to be neglected. The last term alone claims our attention.

Now,

$$\frac{\sin.\,2\,\omega}{u^4}\,\delta.\frac{d\,u}{d\,v} = \frac{\sin.\,2\,\omega}{u^4} \times -\frac{Q\,q}{a}\,\sin.\,q\,v$$

$$= -\,Q\,q\,a^3.\sin.\,(2\,v - 2\,m\,v).\sin.\,q\,v\ \dagger$$

$$= \frac{Q q a^3}{2}\,[\cos.\,(q\,v + 2\,v - 2\,m\,v) - \cos.\,(q\,v - 2\,v + 2\,m\,v)].$$

Hence,

$$\frac{1}{h^2}\,\delta.\left(\frac{T}{u^3}\cdot\frac{d\,v}{d\,v}\right) =$$

---

* See Lacroix, pp. 655, &c.; also Woodhouse's *Calculus of Variations*, p. 82.

† The first term of sin. $2\,\omega$, for reasons before alledged, pp. 167, &c. is sufficient.

$$= \frac{3K}{4h^2} Qq [\cos. (qv - 2v + 2mv) - \cos. (qv + 2v - 2mv)].$$

$$\frac{\mathrm{m}'}{2\mathrm{h}^2} \quad \frac{\mathrm{u}'^3}{\mathrm{u}^3} \quad (\textit{see pp.}\ 153,\ 167.)$$

$$\frac{m'}{2h^2} . u'^3 \delta . \left(\frac{1}{u^3}\right) = - \frac{3m'a^4}{2h^2 a'^3} . \delta u$$

$$= - \frac{3K}{2h^2} . Q \cos. qv.$$

If we do not wish to exclude the correction that involves Qe, then

$$\frac{m'}{2h^2} u'^3 . \delta\left(\frac{1}{u^3}\right) = - \frac{3m'a^3}{2h^2 a'^3} Q \cos. qv\, (1 - 4e . \cos. cv),$$

$$\left(\text{since } \frac{1}{u^4} = a^4 (1 + e \cos. cv)^{-4} = a^4 (1 - 4e \cos. cv), \text{ nearly}\right),$$

consequently that part of the correction which involves Qe is

$$\frac{6K}{h^2} Qe . \cos. qv . \cos. cv,$$

$$\text{or } \frac{3K}{h^2} Qe . [\cos. (qv - cv) + \cos. (qv + cv)].$$

$$\frac{3\mathrm{m}' . \mathrm{u}'^3}{2\mathrm{h}^2 . \mathrm{u}^3} \cos. 2\omega \ (\textit{see pp.}\ 154.\ 167.)$$

$$\delta \left(\frac{\cos. 2\omega}{u^3}\right) = - \frac{3}{u^4} \cos. 2\omega . \delta u + \frac{1}{u^3} \delta . \cos. 2\omega$$

$$= - \frac{3}{u^4} \cos. (2v - 2mv) . Q \cos. qv + \frac{2ma^3 Q}{q} \left\{ \begin{array}{l} \cos. (qv + 2v - 2mv) \\ - \cos. (qv - 2v + 2mv) \end{array} \right\} *$$

$$= - Qa^3 \left(\frac{3}{2} + \frac{2m}{q}\right) \cos. (qv - 2v + 2mv)$$

$$- Qa^3 \left(\frac{3}{2} - \frac{2m}{q}\right) \cos. (qv + 2v - 2mv),$$

---

* See the two last lines of the value of cos. 2 ω, in p. 166.

and this expression multiplied into $\frac{3m'}{2h^2}$ will give us the required correction.

$$\left(\frac{d^2 u}{d v^2} + u\right) \frac{2}{h^2} \int \frac{T\, d v}{u^3} \quad (\textit{see pp.}\ 154.\ 168,\ \&c.)$$

Here we must take the variation by the rule for taking the variation of a rectangle *.

$$\text{1st,} \quad \delta\left(\frac{d^2 u}{d v^2} + u\right) = \frac{d^2 \delta u}{d v^2} + \delta u.$$

$$\text{Now, if} \quad \delta u = \frac{Q}{a} \cos.\, q v;$$

then, (see *Calc. Variations*, p. 84, &c.) $\frac{d^2 \delta u}{d v^2} = -\frac{Q q^2}{a} \cos.\, q v$,

and, consequently,

$$\frac{d^2 \delta u}{d v} + \delta u = \frac{Q}{a}(1 - q^2) . \cos.\, q v.$$

$$\text{2dly,} \quad \frac{T}{u^3} = -\frac{3 m' u'^3}{2 u^4} \sin.\, 2\omega,$$

$$\text{and} \quad \delta\left(\frac{\sin.\, 2\omega}{u^4}\right) = -\frac{4}{u^5} \sin.\, 2\omega \,.\, \delta u + \frac{\delta \,.\, \sin.\, 2\omega}{u^4}$$

$$= -4 Q a^4 . \sin.(2v - 2mv) \cos.\, q v$$

$$+ \frac{2 Q m a^4}{q} \left\{ \begin{array}{l} \sin.\,(q v + 2 v - 2 m v) \\ + \sin.\,(q v - 2 v + 2 m v) \end{array} \right\}$$

$$= Q a^4 . \left(2 + \frac{2m}{q}\right) \sin.\,(q v - 2 v + 2 m v)$$

$$- Q a^4 . \left(2 - \frac{2m}{q}\right) \sin.\,(q v + 2 v - 2 m v).$$

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* See Woodhouse's *Calculus of Variations*, pp. 82, &c.

Hence,

$$\frac{2}{h^2}\delta\int\frac{T\,dv}{u^3} =$$

$$\frac{3\,K\,Q\,a}{h^2}\left\{\begin{array}{l}\dfrac{1}{q-2+2m}\cdot\left(2+\dfrac{2m}{q}\right)\ \text{cos.}\ (qv-2v+2mv) \\ -\dfrac{1}{q+2-2m}\left(2-\dfrac{2m}{q}\right)\ \text{cos.}\ (qv+2v-2mv)\end{array}\right\}$$

Lastly, since $u=\frac{1}{a}\left(1+e^2+\frac{\gamma^2}{4}+e\ \text{cos.}\ cv-\frac{\gamma^2}{4}\ \text{cos.}\ 2gv\right)$,

$$\frac{d^2u}{dv^2}+u=\frac{1}{a}\left(1+e^2+\frac{\gamma^2}{4}+(1-c^2)\,e\,.\,\text{cos.}\,cv+\frac{\gamma^2}{4}\,(4g^2-1)\ \text{cos.}\ 2gv\right).$$

The variation, therefore, of the whole term,

or, $\left(\frac{d^2\,\delta u}{dv^2}+\delta u\right)\frac{2}{h^2}\int\frac{T\,dv}{u^3}+\left(\frac{d^2u}{dv^2}+u\right)\frac{2}{h^2}\ \delta\int\frac{T\,dv}{u^3}$,

will be (see pp. 172, 173.)

$$\frac{Q}{a}(1-q^2)\ \text{cos.}\ qv\times\frac{3Ka}{a}\left\{\begin{array}{l}\dfrac{1+3e^2}{2-2m}\,.\ \text{cos.}\ (2v-2mv) \\ -\dfrac{2\,.\,(1+m)}{2-2m-c}\,e\,.\ \text{cos.}\,(2v-2mv-cv)^{*} \\ -\dfrac{2\,.\,(1-m)}{2-2m+c}\,e\,.\,\text{cos.}\ (2v-2mv+cv) \\ -\ \&c.\end{array}\right\}$$

$$+\frac{3K}{h^2}\left\{\begin{array}{l}\dfrac{Q}{q-2+2m}\left(2+\dfrac{2m}{q}\right)\ \text{cos.}\ (qv-2v+2mv) \\ -\dfrac{Q}{q+2-2m}\left(2-\dfrac{2m}{q}\right)\ \text{cos.}\ (qv+2v-2mv)\end{array}\right\}\times$$

---

* In this and the next term, those terms of $\frac{2}{u^3}\int\frac{T\,dv}{u^3}$ which (see pp. 144, &c.) involve $1-c^2$, are, by reason of their smallness, omitted.

$$\left\{\begin{array}{l} 1 + e^2 + \frac{\gamma^2}{4} + \\ (1 - c^2)\, e \,.\, \cos.\, c v + \\ \frac{4 g^2 - 1}{4} \gamma^2 \cos.\, 2 g v \end{array}\right\}$$

If the terms of this expression are combined according to the formulæ of Trigonometry, and the coefficients of like terms collected together, the whole variation will be nearly equal to (the terms involving $2 e^2$, $2 \gamma^2$ being rejected),

$$\frac{3 K Q}{a_{,}} \left\{\begin{array}{l} \left(\frac{1 - q^2}{4(1-m)} + \frac{2 + \frac{2m}{q}}{q - 2 + 2m}\right) \cos.\, (q v - 2 v + 2 m v) \\ \left(\frac{1 - q^2}{4(1-m)} - \frac{2 - \frac{2m}{q}}{q + 2 - 2m}\right) \cos.\, (q v + 2 v - 2 m v) \end{array}\right\}$$

$$+ \frac{3 K}{a_{,}} Q e (q^2 - 1) \frac{1 + m}{2 - 2m - c} \cos.\, (q v - 2 v + 2 m v + c v)^{*}$$

$$+ \frac{3 K}{a_{,}} Q e (q^2 - 1) \frac{1 - m}{2 - 2m + c} \cos.\, (q v - 2 v + 2 m v - c v)$$

$$+ \,\&c.$$

We will now exhibit, under one view, in a Table, and for the purpose of reference, the several corrections of Π, as they are respectively due to the variations $\delta u'$, δu.

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* These are the principal terms that involve $2 e$: other terms involving that rectangle would arise (see p. 172.) by substituting for u^{-5} and u^{-4} not a^5 and a^4, but $a^5 (1 - 5 e \cos.\, c v)$ and $a^4 . (1 - 4 e \cos.\, c v)$; and, by taking into account those terms in $\delta \,.\, \sin.\, 2 \omega$ which involve $2 e$. But the combinations involving such terms may be neglected since they contain $2 m e$, the product of three small quantities. The terms which involve the cosines of $q v + 2 v - 2 m$, $q v + 2 v - 2 m v \pm c v$, as being of no use, are also omitted.

Table of Corrections.

$$\delta u' = \frac{e'}{a'} \cos. c' m v.$$

Terms.		Corrections.
$\frac{1}{h^2} \cdot \frac{T}{u^3} \cdot \frac{du}{dv}$	p. 167.	0
$\frac{m'}{2h^2} \cdot \frac{u'^3}{u^3}$	p. 167.	$\frac{3K}{2a_{,}} e' . \cos. c' m v$
$\frac{3m'}{2h^2} \cdot \frac{u'^3}{u^3} \cos. 2\omega,$	p. 168.	$\frac{3K}{2a_{,}} e' \left\{ \begin{array}{l} \frac{7}{2} \cos. (2v - 2mv - c' m v) \\ - \frac{1}{2} \cos. (2v - 2mv + c' mv) \end{array} \right\}$
$\left(\frac{d^2u}{dv^2} + u \right) \frac{2}{h^2} \int \frac{T dv}{u^3},$	p. 168.	$\frac{3K}{2a_{,}} e' \left\{ \begin{array}{l} 7 . \frac{\cos. (2v - 2mv - c' m v)}{2 - 2m - c' m} \\ - \frac{\cos. (2v - 2mv + c' m v)}{2 - 2m + c' m} \end{array} \right\}$

$$\delta u = \frac{Q}{a} \cos. q v.$$

Terms.		Corrections.
$\frac{1}{h^2} \cdot \frac{T}{u^3} \cdot \frac{du}{du}$	p. 170.	$\frac{3K}{4h^2} . Q q \left\{ \begin{array}{l} \cos. (qv - 2v + 2mv) \\ - \cos. (qv + 2v - 2mv) \end{array} \right\}$
$\frac{m'}{2h^2} \cdot \frac{u'^3}{u^3},$	p. 171.	$- \frac{3K}{2h^2} Q \cos. q v$
$\frac{3m'}{2h^2} \cdot \frac{u'^3}{u^3} \cos. 2\omega,$	p. 171.	$- \frac{3KQ}{2h^2} \left\{ \begin{array}{l} \left(\frac{3}{2} + \frac{2m}{q} \right) \cos. (qv - 2v + 2mv) \\ + \left(\frac{3}{2} - \frac{2m}{q} \right) \cos. (qv + 2v - 2mv) \end{array} \right\}$

Continuation of Table of Corrections;

the term being

$$\left(\frac{d^2u}{dv^2}+u\right)\frac{2}{h^2}\int\frac{T\,dv}{u^3}, \quad \text{p. 173.}$$

the correction is

$$\frac{3KQ}{a}\left\{\begin{array}{l} \left(\dfrac{1-q^2}{4.(1-m)}+\dfrac{2+\dfrac{2m}{q}}{q-2+2m}\right)\cos.(qv-2v+2mv) \\ +\dfrac{(q^2-1)(1+m)}{2-2m-c}e.\cos.(qv-2v+2mv+cv) \\ +\dfrac{(q^2-1)(1-m)}{2-2m+c}e.\cos.(qv-2v+2mv-cv) \\ +\&c. \end{array}\right\}$$

The last part in the preceding Table originates entirely from the variation $\delta u=\frac{Q}{a}\cos.qv$: now, $\frac{Q}{a}\cos.qv$ is an assumed term, with this condition alone, that the coefficient Q is a very small quantity. $\frac{Q}{a}\cos.qv$, therefore, may represent any of the small terms in the value of u, whether such term be a deficient or suppressed term (suppressed for the purpose of rendering the conditions of the problem more simple), or an additional term acquired by approximation and the integration of the differential equation. For instance, in the first case, the equation for u having been assumed (see pp. 150. 155.)

$$u=\frac{1}{a}(1+e^2+e.\cos.cv),$$

and the real equation being (see p. 162.),

$$u=\frac{1}{a}\left(1+e^2+\frac{\gamma^2}{4}+e.\cos.cv-\frac{\gamma^2}{4}\cos.2gv\right)$$

$\frac{\gamma^2}{4a}$, and $-\frac{\gamma^2}{4}\cos.2gv$ are *suppressed* or *deficient* terms. In order then to supply the corrections that are due to their omission, we have nothing else to do than to find from the Table of p. 175. the corrections that respectively arise when $\frac{Q}{a}\cos.qv$ is made to represent $\frac{\gamma^2}{4}$ and $-\frac{\gamma^2}{4}\cos.2gv$. In the first case, then,

$$Q \text{ must} = \frac{\gamma^2}{4} \text{ and } q = 0.$$

In the second, Q must $= -\frac{\gamma^2}{4}$ and $q = 2g$.

The following Table will exhibit the results (see Table, p. 175.)

	Hypothesis. $Q = \frac{\gamma^2}{4},\ q = 0.$
Terms.	Corrections.
$\frac{1}{h^2}\cdot\frac{T}{u^3}\cdot\frac{du}{dv}$	0
$\frac{m'}{2h^2}\cdot\frac{u'^3}{u^3}$	$-\frac{3K}{2h^2}\ \frac{\gamma^2}{4}$
$3m'\cdot\frac{u'^3}{u^3}\cos.2\omega$	$-\frac{9K}{2h^2}\cdot\frac{\gamma^2}{4}\cos.2\omega$
$\left(\frac{d^2u}{dv^2}+u\right)\frac{2}{h^2}\int\frac{T\,dv}{u^3}$	$-\frac{9K}{8h^2}\cdot\frac{\gamma^2}{1-m}\cos.2\omega$

Hypothesis.

$$Q = -\frac{\gamma^2}{4},\ q = 2g.$$

Terms.	Corrections.
$\frac{1}{h^2}\cdot\frac{T}{u^3}\cdot\frac{du}{dv}$	$-\frac{3K}{4h^2}\cdot\frac{g\gamma^2}{2}\left\{\begin{array}{l}\cos.(2gv-2v+2mv)\\ -\cos.(2gv+2v-2mv)\end{array}\right\}$
$\frac{m'}{2h^2}\cdot\frac{u'^3}{u^3}$	$\frac{3K}{2h^2}\cdot\frac{\gamma^2}{4}\cdot\cos.2gv$
$3m'\cdot\frac{u'^3}{u^3}\cos.2\omega$	$\frac{3K}{2h^2}\cdot\frac{\gamma^2}{4}\left\{\begin{array}{l}\left(\frac{3}{2}+\frac{m}{g}\right)\cos.(2gv-2v+2mv)\\ +\left(\frac{3}{2}-\frac{m}{g}\right)\cos.(2gv+2v-2mv)\end{array}\right\}$
$\left(\frac{d^2u}{dv^2}+u\right)\times\frac{2}{h^2}\int\frac{Tdv}{u^3}$	$\frac{3K}{h^2}\gamma^2\left\{\begin{array}{l}\left(\frac{4g^2-1}{16(1-m)}-\frac{2+\frac{m}{g}}{4.(2g-2+2m)}\right)\cos.(2gv-2v+2mv)\\ \left(+\frac{4g^2-1}{16(1-m)}-\frac{2-\frac{m}{g}}{4.(2g+2-2m)}\right)\cos.(2gv+2v-2mv)\end{array}\right\}$

By means of these corrections and the previous ones of pp. 163, 175, &c. we may supply the deficiencies of the terms composing Π and add its new terms. But we must add another correction on account of the variation of h^2, which $= a_{,}(1-e^2-\gamma^2)$. Now every term of Π (see pp. 169, &c.) is divided by h^2, and since $\frac{1}{h^2} = \frac{1}{a_{,}(1-e^2-\gamma^2)} = \frac{1}{a_{,}}(1+e^2+\gamma^2)$, nearly, Π will become by the variation of h^2, $\Pi + \Pi\frac{\gamma^2}{a_{,}}$, nearly; the value, therefore, of $\frac{m'u'^3}{2h^2u^3}$ will become

$$\frac{K}{2a_{,}}\cdot(1+e^2+\gamma^2-3e\cdot\cos.cv+\&c.)$$

rejecting the terms involving $\epsilon\,\gamma^2$, &c.: the alteration therefore of $\frac{m'\,u'^3}{2\,h^2\,u^3}$ will take place solely in its constant part, which will become

$$\frac{K}{2\,a_{,}}\cdot(1+e^2+\gamma^2).$$

But by the preceding Table the correction in this term, from $\delta\,u=\frac{\gamma^2}{4}$, is $-\frac{3\,K}{2\,a_{,}}\cdot\frac{\gamma^2}{4}$: the whole term therefore resulting from the two corrections (both originating from the introduction of the condition of the inclination) is

$$\frac{3\,K}{2\,a_{,}}\left(1+e^2+\gamma^2-\frac{3\,\gamma^2}{4}\right)=\frac{3\,K}{2\,a_{,}}\left(1+e^2+\frac{\gamma^2}{4}\right).$$

Again, the coefficient of $\cos.(2\,v-2\,m\,v)$, in the term $\frac{3\,m'}{2\,h^2}\,\frac{u'^3}{u^3}\cos.\,2\,\omega$, will, by reason of the variation of h^2 (see p. 178.) and correction due to $\frac{\gamma^2}{4}$ (see Table, p. 177.) become

$$\frac{3\,K}{2\,a_{,}}\left(1+e^2+\gamma^2-\frac{3\,\gamma^2}{4}\right)=\frac{3\,K}{2\,a_{,}}\left(1+e^2+\frac{\gamma^2}{4}\right).$$

Lastly, the coefficient of $\cos.(2\,v-2\,m\,v)$ in the term $\frac{2}{h^2}\int\frac{T\,d\,v}{u^3}$ instead of being $\frac{3\,K}{2\,a_{,}}\cdot\frac{1+2\,e^2}{1-m}$ will become by the variation of h^2 (see p. 178.)

$$\frac{3\,K}{2\,a_{,}}\cdot\frac{1+2\,e^2+\gamma^2}{1-m},$$

and, by the correction of the preceding Table, (see p. 177.)

$$\frac{3\,K}{2\,a_{,}}\cdot\frac{1+2\,e^2+\gamma^2}{1-m}-\frac{3\,K}{2\,a_{,}}\cdot\frac{3\,\gamma^2}{4},$$

$$\text{or }\frac{3\,K}{2\,a_{,}}\cdot\left\{\frac{1+2\,e^2+\frac{\gamma^2}{4}}{1-m}\right\},\text{ nearly.}$$

If we now collect these corrections, we may exhibit the differential equation under the following form (see pp. 144. 156.)

$$0=\frac{d^2 u}{d v^2}+u-\frac{1}{a_{,}}\left(1+e^2+\frac{\gamma^2}{4}\right)+\frac{K}{2a_{,}}\cdot\left(1+e^2+\frac{\gamma^2}{4}\right)$$

$$-\frac{3K}{2a_{,}}\cdot\left(1+\frac{e^2}{2}\right)e\,.\cos.\,cv+\frac{3K}{2a_{,}}e^2\,.\cos.\,2cv$$

$$+\frac{3K}{2a_{,}}\left(1+{}^{*}(1+2m)e^2+\frac{\gamma^2}{4}+\frac{1+3e^2+\frac{\gamma^2}{4}}{1-m}\right)\cos.\,(2v-2mv)$$

$$+\frac{3K}{a_{,}}\left(\frac{c}{4}-\frac{3+4m}{4}-\frac{2\,.(1+m)}{2-2m-c}+\frac{1-c^2}{4(1-m)}\right)e\cos.(2v-2mv-cv)$$

$$-\frac{3K}{a_{,}}\left(\frac{c}{4}+\frac{3-4m}{4}+\frac{2\,.(1-m)}{2-2m+c}+\frac{1-c^2}{4(1-m)}\right)e\,.\cos.\,(2v-2mv+cv)$$

$$+\frac{3K}{4a_{,}}\left(\frac{2+11m}{2}-\frac{10+19m}{2c-2+2m}\right)e^2\,.\cos.\,(2cv-2v+2mv)$$

$$-\frac{3K}{4a_{,}}\left(\frac{4-2m+c'm}{2-2m+c'm}\right)e'\,.\cos.(2v-2mv+c'mv)$$

$$+\frac{3K}{4a_{,}}\left(\frac{7\,.(4-2m-c'm)}{2-2m-c'm}\right)e'\,.\cos.\,(2v-2mv-c'mv)$$

$$+\frac{3K}{2a'}e'\cos.\,(c'mv)$$

$$-\frac{3}{4a_{,}}\left(1+e^2-\frac{K}{2}\right)\gamma^2\cos.\,2gv$$

$$+\frac{3K}{4a_{,}}\left(\frac{3+2m-2g^2}{4g}+\frac{4g^2-1}{4(1-m)}-\frac{2+\frac{m}{g}}{2g-2+2m}\right)\times$$
$$\gamma^2\,.\cos.\,(2gv-2v+2mv)$$

$$+\frac{3K}{4a_{,}}\left(\frac{3-2m+2g^2}{4g}+\frac{4g^2-1}{4.(1-m)}-\frac{2-\frac{m}{g}}{2g+2-2m}\right)\times$$
$$\gamma^2\cos.\,(2gv+2v-2mv).$$

$$+\ \&c.$$

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* $1+2mc=1+2m$, nearly.

If we compare this equation with the former one of p. 156. we shall see that the conditions of the eccentricity of the Solar Orbit, and the inclination of the plane of the Lunar Orbit to that of the ecliptic, introduce six additional terms to the value of u, even when the terms involving $e\,e'$, $e\,\gamma^2$, $e'\,\gamma^2$, &c. are excluded from the result.

The coefficient of cos. $c\,v$ remains the same as it was in p. 157. The error therefore in c, and consequently in the progression of the apogee, is not, in the slightest degree, lessened by this second approximation; and, it is easy to see that it is quite hopeless to expect the correction of the error from that kind of approximation which has hitherto been used, and which consists in successively taking account of small quantities rejected in a previous process.

It is not therefore to be wondered at, that a panic should have seized Clairaut and the mathematicians who had adopted Newton's system, when, on the first revision of their calculations, they could discover no source of error to which so large an one as that in the computed progression could be traced.

Since the terms involving e'^2, &c. have been purposely excluded, the coefficient of cos. $c\,v$ does not contain that quantity. If retained, it would, very inconsiderably, affect the numerical value of the coefficient, and would in no wise relieve it from the error it labours under with regard to the quantity of the progression. But, for other purposes, it is quite essential to retain it, since it enables us to explain, on theoretical principles, the *secular equation* of the progression of the Lunar Apogee.

That the coefficient of cos. $c\,v$ will contain e'^2 may immediately be seen by expanding $\frac{m'\,u'^3}{2h^2u^3}$: now

$$\frac{m'\,u'^3}{2h^2u^3} = \frac{m'}{2\,h^2}\,\frac{a^3}{a'^3}\,(1 + e'\,.\,\text{cos.}\,c'\,v)^3\,(1 + e^2 + e\,.\,\text{cos.}\,c\,v)^{-3}$$

$$= \frac{m'}{2\,h^2}\,\frac{a^3}{a'^3}\left(1 + \frac{3\,e'^2}{2} + \&\text{c.}\right)(1 - 3\,e\,\text{cos.}\,c\,v + \&\text{c.})$$

consequently, one term in the coefficient of cos. $c\,v$ will be

$$-\frac{9K}{4h^2}\cdot e'^2 e,$$

which, since e' is subject to alteration, will give rise to an alteration in the coefficient of cos. cv: consequently c, on which the progression depends, will not always result of the same value.

In like manner, if we retain the terms involving e'^2, the constant part of u, as resulting from the integration of the differential equation, will contain a term $= \frac{3K}{4a} e'^2$ (see pp. 159, &c.) a term, if we regard its numerical value, of no importance, (since it never exceeds .0000012); but, on account of the *variability* of e', of considerable moment, since it is the exponent of the *secular* equation of the Moon's mean longitude (see *Astronomy*, p. 312.)

We have now explained the principle and the method of successively correcting the results in the Problem of the Three Bodies. The eccentricities of the Solar and Lunar Orbits, and the inclinations of their planes have been taken account of; and the hypothetical conditions of the problem have been made to approach their real state in nature. All material causes of error, therefore, in these respects are rescinded. Still, from what has been just said (pp. 181, &c.), if Newton's system be true, the preceding processes stand in need of a farther correction. And this is the case: but the correction that remains to be made, is of a kind totally dissimilar to the preceding corrections. It refers not to the supplying of any omitted or deficient condition, but to the very principle of that computation by which the value of u is determined.

If we refer to pp. 156. 180. it will be seen that the value of u is determined by the *approximate* integration of the differential equation

$$\frac{d^2u}{dv^2} + u + \Pi = 0,$$

for, not only are several terms in the expanded expression for Π not retained, but the expanded expression must be imperfect, since (see pp. 150, &c.) it is procured by substituting in Π, which is a function of u, an approximate and imperfect value of u. That first assumed value is, as it has been already stated, the *elliptical* value of u, which cannot subsist with the hypothesis of a disturbing force: and if there were required any farther proof of the neceſsary imperfection of the method, we have only to compare the assumed value of u with its *resulting* value: between which values *a difference*, *at least*, exists.

It may, however, be said generally that the imperfection of the method for determining u is of that kind which belongs to every method of approximation, and which may, in the usual manner, be remedied. With the last acquired value the whole computation should be repeated. This method, however, in the present instance would be very tedious. We will endeavour, therefore, in the next Chapter, to attain the same end by a shorter route: by applying, in fact, the principle and formulæ of the preceding corrections.

CHAP. XII.

Principle of the Method of correcting the Value of the Radius Vector, obtained by an Approximate Integration of the Differential Equation.

THE general equation,

$$\frac{d^2 u}{d v^2} + u - \frac{\mu}{h^2} - \Omega = 0,$$

where Ω (see pp. 98, &c.) represents the disturbing force, cannot generally be solved. In order to approximate to its solution we assume that value of u which is the integral of the equation when $\Omega = 0$; which value, in other words, is the *elliptical* value of u, and the true value when no disturbing force acts. This value is substituted in Ω $\left(\Pi = -\frac{\mu}{h^2} - \Omega\right)$, the equation integrated, and a new value of u obtained; which, since the conditions of the problem are rightly involved in the general differential equation, must be more nearly the true value than the one assumed.

Still it is not the true value: in order more nearly to approach to it, we may substitute the last obtained value in Π, and again integrate the resulting equation. Now if we attend to the process of pp. 169, &c. we shall find that its effect is to add several small terms to the first assumed value of u. Suppose, (for the sake of stating the case in the most simple manner), that the first integration adds one small term to the value of u: then, if the process be repeated with this augmented value of u, Π (see pp. 170, &c.) will contain more terms than it did before; which additional terms are entirely due to the augmentation of u: they may be viewed, therefore, as so many *corrections* to its value: and, accordingly, we need only compute the corrections to the value of Π. Now this we can do by the Table already formed (see p. 175.), for $\frac{Q}{a}$ cos. $q\,v$ may represent any term: either a *deficient* one in the *elliptical* value of u, or an *additional one acquired by integration.*

We have supposed the value of u to contain, after integration, one additional term: the fact is, it will contain several. The additional terms then in Π will be corrections due to the additional terms of u: but, since these latter are, in the cases treated of, very small, we may deduce the corrections separately, one by one; and $\frac{Q}{a}$ cos. $q\,v$ which may represent any term, will thus serve, by repetition of process, to represent all.

This is a brief description of the principle of the method, which we will now exemplify.

The first principal additional term in the differential equation is (see p. 180.)

$$\frac{3\,K}{2\,a_{\backslash}}\left\{1+(1+2m)\,e^2+\frac{\gamma^2}{4}+\frac{1+3\,e^2+\frac{\gamma^2}{4}}{1-m}\right\}\text{cos. }(2v-2\,m\,v).$$

If (see p. 100.) we divide this term by $(2-2\,m)^2-1$, then the result is an additional term in the value of u, or is a correction to its first assumed and elliptical value; and, if we equate it with $\frac{Q}{a}$ cos. $q\,v$, or $\frac{Q}{a_{\backslash}}$ cos. $q\,v$ (since a and $a_{\backslash}$ are nearly equal), we shall have

$$Q\,[4.(1-m)^2-1]=\frac{3\,K}{2}\left(1+(1+2m)\,e^2+\frac{\gamma^2}{4}+\frac{1+3\,e^2+\frac{\gamma^2}{4}}{1-m}\right),$$

and $q=2-2\,m.$

In order therefore to find what new terms will be added to Π, or what corrections to existing terms, we must, in the Table of p. 175. substitute for Q its preceding value, and $2-2\,m$ for q.

The results then will be (see pp. 175, 176.), respectively,

$$\frac{3\,K}{4\,h^2}\,Q\,.\,(2\,-\,2\,m)\,[\cos.\;0\,v - \cos.\;(4\,v\,-\,4\,m\,v)],$$

$$-\,\frac{3\,K}{2\,h^2}\,.\;Q\cos.\;(2v - 2\,m\,v),$$

$$-\,\frac{3KQ}{2\,h^2}\left[\left(\frac{3}{2}+\frac{m}{1-m}\right)\cos.\;0\,v + \left(\frac{3}{2}-\frac{m}{1-m}\right)\cos.\;(4\,v\,-\,4\,m\,v)\right],$$

$$+\,\frac{3KQ}{a_{\backslash}}\left\{\begin{array}{l}\dfrac{1\,-\,4\,(1\,-\,m)^2}{4\,.\,(1\,-\,m)}\cos.\;0\,v\\[2ex] +\,[4(1-m)^2-1]\left(\dfrac{1+m}{2-2m-c}+\dfrac{1-m}{2-2m+c}\right)e\cos.\;cv\end{array}\right\}$$

Hence, there will be only one new term added to the value of Π, the *argument* of which will be $4\,v\,-\,4\,m\,v$, and the term itself will be

$$-\,\frac{3\,KQ}{2\,h^2}\left(1\,-\,m\,+\,\frac{3}{2}\,-\,\frac{m}{1-m}\right)\cos.\;(4\,v\,-\,4\,m\,v)\,;$$

the corresponding additional term in the value of u, after integration, will be the preceding term divided by $16\,.\,(1-m)^2\,-\,1$.

The other parts are corrections of terms obtained by the first approximation and integration: first, since $\cos.\;0\,v = 1$, the correction of the constant part of Π will be (see lines 1, 3, 4, of this page),

$$\frac{3\,KQ}{a_{\backslash}}\left(\frac{1\,-\,m}{2}\,-\,\frac{3}{4}\,-\,\frac{m}{2\,(1\,-\,m)}\,+\,\frac{1\,-\,4\,.\,(1\,-\,m)^2}{4\,.\,(1-m)}\right)$$

$$=\,-\,\frac{3\,K\,Q}{a_{\backslash}}\left(\frac{4\,-\,m\,+\,m^2}{4}\right),\ \text{nearly},$$

and the corresponding correction that would be given to u, after integration, is (see p. 100.) the preceding term with the sign changed.

Secondly, the term in the second line of this page, namely,

$$-\,\frac{3\,K}{2\,h^2}\,Q\,\cdot\,\cos.\;(2\,v\,-\,2\,m\,v),$$

is the correction of a term with the same argument already existing in Π: and the corresponding correction to the value of u, resulting from integration, is the preceding correction divided by $4.(1-m)^2-1$.

Lastly, the term

$$\frac{3\,KQ}{a_{,}}\left\{[4.(1-m)^2-1]\left(\frac{1+m}{2-2m-c}+\frac{1-m}{2-2m+c}\right)\right\}e.\cos.\,c\,v,$$

is the correction of

$$-\frac{3\,K}{2\,a_{,}}\left(1+\frac{e^2}{2}\right)e\,.\cos.\,c\,v,$$

in the differential equation of p. 180: so that, if we were now to apply the correction, the coefficient of cos. $c\,v$ would become

$$-\frac{3K}{a_{,}}\left\{\frac{1}{2}+\frac{e^2}{4}-[4.(1-m)^2-1].\left(\frac{1+m}{2-2m-c}+\frac{1-m}{2-2m+c}\right)Q\right\}e,$$

and thus we may perceive that the correction due even to one additional term, has made a considerable alteration in the coefficient of that important term on which the progression of the apogee depends. But the coefficient will receive corrections from other additional terms.

For, as (see pp. 170, 176, &c.) $\frac{Q}{a}$ cos. qv may represent any term in the value of u, let us suppose the differential equation of p. 180. to be integrated, and that $\frac{Q}{a}$ cos. $q\,v$ represents the term whose argument is the same as that of the second principal additional term of the equation: then, nearly,

$$Q=\frac{3\,K\,e}{(2-2\,m-c)^2-1}\left(\frac{c}{4}-\frac{3+4\,m}{4}-\frac{2\,(1+m)}{2-2m-c}+\frac{1-c^2}{4\,(1-m)}\right),$$

and $q=2-2\,m-c$.

If this value of q be substituted in the Table p. 175. the arguments of the resulting corrections will be

$$c\,v \qquad ,\ 4\,v-4\,m\,v-c\,v,$$

$$2v - 2mv - cv,$$

$$cv \qquad , 4v - 4mv - cv,$$

$$cv$$

$$0v$$

$$2cv,$$

and of these corrections, there is only one that of which the argument is $4v-4mv-cv$), which after a second integration, will produce a new term; the others are corrections of terms already obtained by the first integration. Now of these latter (and this is a point on which the determination of the progression of the Lunar Apogee depends) three have the argument cv, or serve to correct the coefficient of cos. cv: and their sum is

$$\frac{3K}{4a_{\backslash}}Q\left\{\begin{array}{l}(2-2m-c)-\left(3+\dfrac{4m}{2-2m-c}+\dfrac{1-(2-2m-c)^2}{1-m}\right.\\ \qquad\qquad -\dfrac{8}{c}-\dfrac{8m}{c\,.\,(2-2m-c)}\end{array}\right\}$$

If we take $\frac{Q}{a}$ cos. qv to represent the term which would be added to the value of u, after the integration of the differential equation, and in consequence of that term therein contained, which has for its argument $2v - 2mv + cv$, then, as in the former case, there will result three corrections to the coefficient of cos. cv, and their sum will be

$$\frac{3K}{4a_{\backslash}}Q\,.\left\{\begin{array}{l}2-2m+c-\left(3+\dfrac{4m}{2-2m+c}\right)+\dfrac{1-(2-2m+c)^2}{1-m}\\ \qquad\qquad +\dfrac{8}{c}+\dfrac{8m}{c\,.\,2-2m+c}\end{array}\right\}$$

in which expression, Q must equal the coefficient of

$$\cos.\ (2v - 2mv + cv),$$

(in the differential equation of p. 180), divided by

$$(2-2m+c)^2 - 1.$$

We may conclude then from what has preceded, that the terms

$$A \,.\, \cos.\,(2\,v - 2\,m\,v),\quad B \,.\, \cos.\,(2\,v - 2\,m\,v - c\,v),$$

$C \,.\, \cos.\,(2\,v - 2\,m\,v + c\,v)$, resulting from the first integration of the differential equation, would, by the repetition of the processes of approximation and integration, chiefly serve to correct the coefficient of $\cos.\,c\,v$. This has been established by the very result of the process of Correction; but it is easy to perceive from an inspection of the Table of p. 175, that it must happen.

This is one important fact: another curious fact, in the preceding system of Corrections, is to be noted in the Correction which each term confers on itself. Thus, the term being $\frac{Q}{a}\cos.\,q\,v$, the second correction of the Table of p. 175. is

$$-\frac{3\,KQ}{2\,h^2}\cos.\,q\,v\,;$$

but (see p. 185.) that is the correction of Π, and consequently the corresponding correction in the value of u resulting from integration will be

$$\frac{3\,K\,Q}{2\,h^2\,(1-q^2)}\cos.\,q\,v\,;$$

but this being viewed as a new term in the value of u, the correction of the term involving $\cos.\,q\,v$ in Π will be

$$-\frac{3\,K}{2\,h^2}\times\frac{3K\,Q}{2\,h^2\,(1-q^2)}\,a\cos.\,q\,v,$$

and the correction of the like term in the value of u resulting from a repeated integration will be (see p. 185, &c.),

$$-\frac{3\,K}{2\,h^2}\,.\,\frac{3\,K\,Q}{2\,h^2\,.\,(1-q^2)\,(q^2-1)}\,a\cos.\,q\,v,$$

$$\text{or, } \left(\frac{3\,K}{2\,h^2\,(1-q^2)}\right)^2 .\ Q\,a\,.\cos.\,q\,v,$$

and so on. Hence, if $\frac{Q}{a}\cos.\,q\,v$ represent an additional term introduced by the approximate integration of the differential equation, the more correct value of that term will be

$$\frac{Q}{a}\left[1+\frac{3Ka}{2h^2(1-q^2)}+\left(\frac{3Ka}{2h^2(1-q^2)}\right)^2+\text{\&c.}\right]\cos.qv;$$

but the series of terms within the brackets is a geometrical series, and accordingly if we make $\alpha=\frac{3Ka}{2h^2.(1-q^2)}=$ (nearly), $\frac{3Ka}{2a_{,}(1-q^2)}=$ (since $a=a_{,}$ nearly,) $\frac{3K}{2.(1-q^2)}$, we shall have its sum nearly equal to $\frac{1}{1-\alpha}$, and the corrected value of the term will be

$$\frac{Q}{a.(1-\alpha)}\cos.qv.$$

This expression represents the term involving $\cos.qv$ together with the whole series of corrections *derived from itself:* but the term is affected with other, besides the latter, corrections, although less important ones. The series of corrections is, in fact, interminable: for, every new term is a source of corrections which may be viewed as terms, and which, in that character, will give rise to ulterior corrections.

The term corresponding to $\frac{Q}{a}\cos.qv$, in the differential equation, is $\frac{Q}{a}(q^2-1).\cos.qv$: and, for the same reason, the *corrected* term in the differential equation corresponding to the corrected term $\frac{Q}{a.(1-\alpha)}\cos.qv$ is $\frac{Q}{a}.\frac{q^2-1}{1-\alpha}\cos.qv$: hence, if $\frac{3K}{a}.P.\cos.pv$ be a term in the differential equation, $\frac{3K}{a}.\frac{P}{1-\alpha}\cos.pv$, $\left(\alpha=\frac{3K}{2(1-p^2)}\right)$ is the corrected term: let then P', P'', P''', &c. represent those parts of the coefficients of $\cos.(2v-2mv)$, $\cos.(2v-2mv-cv)$, $\cos.(2v-2mv+cv)$, that are within the brackets (see p. 180.) the differential equation, with its *partially* corrected coefficients, will be

$$\begin{aligned} &\frac{d^2 u}{d v^2} + u - \&c. \\ &\quad + \frac{3K}{2a_{,}} \cdot \frac{P'}{1-a'} \cdot \cos.(2v - 2mv) \\ &\quad + \frac{3K}{a_{,}} \cdot \frac{P''}{1-a''}\, e \cos.(2v - 2mv - cv) \\ &\quad - \frac{3K}{a_{,}} \cdot \frac{P'''}{1-a'''}\, e \cos.(2v - 2mv + cv) \\ &\quad + \&c. \end{aligned}$$

in which

$$a' = \frac{3K}{2\cdot[1 - 4\cdot(1-m)^2]},$$

$$a'' = \frac{3K}{2\cdot[1 - (2 - 2m - c)^2]},$$

$$a''' = \frac{3K}{2\cdot[1 - (2 - 2m + c)^2]}.$$

These expressions for the coefficients are very convenient in computation, and give, very nearly, their true values; but not exactly so, since they embrace only the *self-derived* corrections. Now a term must serve to correct, besides itself, other terms. For instance, the term $\frac{Q}{a}\cos.(2v - 2mv)$ will, by combining with the term $\frac{Q'}{a}\, e \cdot \cos.(2v - 2mv \mp cv)$, produce a correction of the term involving $\cos. cv$, (as may be seen in pp. 185. 187.); but this happens only when great exactness is required; for, the coefficient of the correction must involve the product of two small quantities, Q, for instance, and e. In like manner, if from the corrections of the term $\frac{m'}{2h^2}\,\frac{u'^3}{u^3}$ (see pp. 177. 178.) we do not exclude the corrections that involve the products of small quantities, there will arise, besides those we have stated, other corrections to the coefficients of $\cos.(2v - 2mv \mp cv)$; for, since (see this page),

$$\frac{3K}{2a_{,}} \cdot \frac{P'}{(1-a')\cdot[4\cdot(1-m)^2 - 1)} = -\frac{Pa'}{1-a'},$$

is the coefficient of cos. $(2v-2mv)$ in the equation which is the integral of the preceding differential equation (p. 180: l. 3.), it will be what Q represents in p. 185. and, accordingly, the correction derived from it will be

$$\frac{3K}{h^2}\;\frac{P'a'}{a'-1}\,.\,e\,[\text{cos.}\,(2v-2mv-cv)+\text{cos.}\,(2v-2mv+cv)],$$

the terms, therefore, in the third and fourth lines of the preceding equation, will become

$$\frac{3K}{a_{\prime}}\,.\left(\frac{P''}{1-a''}-\frac{a'P'}{1-a'}\right)e\,.\,\text{cos.}\,(2v-2mv-cv)$$

$$-\frac{3K}{a_{\prime}}\,.\left(\frac{P'''}{1-a'''}+\frac{a'P'}{1-a'}\right)e\,.\,\text{cos.}\,(2v-2mv+cv).$$

If we refer to the Table of p. 175. it will be seen that the terms involving e', &c. require corrections similar to the preceding. Thus,

$$\frac{3K}{4a_{\prime}}\,.\,Se'\,.\,\text{cos.}\,(2v-2mv-c'mv),$$

$$-\frac{3K}{4a_{\prime}}\,.\,Te'\,.\,\text{cos.}\,(2v-2mv+c'mv),$$

being two terms of the value of Π, or, which is the same, two terms in the differential equation, their corrections derived from themselves will be (see Table, p. 175.)

$$-\frac{3K}{2a_{\prime}}\times\frac{3KSe'}{2\,.\,[(2-2m-c'm)^2-1]}\,\text{cos.}\,(2v-2mv-c'mv),$$

and

$$\frac{3K}{2a_{\prime}}\,.\,\frac{3KTe'}{2\,.\,[(2-2m+c'm)^2-1]}\,\text{cos.}\,(2v-2mv+c'mv),$$

and consequently the corrected coefficients will be

$$\frac{3Ke'}{4a_{\prime}}\,.\left(S-\frac{3KS}{2\,.\,[(2-2m-c'm)^2-1]}\right),$$

and

$$-\frac{3Ke'}{4a_{\prime}}\,.\left(T+\frac{3KT}{2\,.\,[(2-2m+c'm)^2-1]}\right).$$

We have now, almost enough for exactness, and certainly with sufficient fullness for the elucidation of method, deduced the several terms, and their corrections, of the differential equation, from which, by a previously established process of integration, (see pp. 99, &c.), u may be deduced. It is chiefly in the Lunar Theory that great accuracy is required: not that the determination of the Moon's place differs essentially, or in the analytical mode of treating it, from the determination of Venus's place disturbed by the Earth's action: for, both cases equally belong to the Problem of the Three Bodies. But, the Moon's irregularities carefully observed during a long series of years, and, from the circumstance of her proximity to the Earth, noted with superior exactness, furnish a surer and more eminent test of the truth of Newton's System, than the irregularities of any other planet. The test consists in the comparison of the Moon's computed with her observed place; if the one be accurately noted, the other must be scrupulously computed. The computation, however, after all, must be one of approximation. Some quantities must be rejected, and since by the operation of that peculiar process which is used (see pp. 162, &c.) the values of quantities are continually changing, there can be no general rule, founded on their mere minuteness, for the rejection of some and the retention of others. We cannot be sure of being correct by any method that is much short of actual trial.

But, if we could get rid of this class of difficulties, we should still have to contend with another arising from the necessary complication of the conditions of the problem The disturbance of the Elliptical System is no other than that of all its laws; and consequently it is their analytical expression which is subject to change. In the value of Π, for instance, a term occurs (see p. 169.) $\frac{-3}{2} u^2 s^2$, and s^2 was assumed equal $\gamma^2 \sin.^2 gv$; an assumption, in principle, not compatible with the existence of a disturbing force. Instead of s^2 we ought to have assumed $(s + \delta s)$ in which δs should be supposed to represent a variation of s arising from the disturbing force. And this assumption would have introduced $\frac{3 s \delta s}{h^2}$ into the value of Π. But δs, representing the variation of s from its elliptical value, (or rather its value in the undisturbed system) can only be known by the integration of the third equation,

$$\frac{d^2 s}{dv^2} + s + \Sigma = 0.$$

If therefore our object were scrupulously to compute the value of u, it would be necessary, after the approximations already pointed out to be made, to obtain, by the approximate integration of the third equation *, the value of s and to substitute it in the first equation.

In order, therefore, steadily to pursue the obvious method of successive corrections, it is necessary to deduce δs from the third equation, to substitute its value in the first, and then to deduce the value of u. But we shall be content, at present, with having pointed out the source of this new correction, of which however the detail and application, since it is small in degree, would not be very tedious. The design and scope of this Treatise call our attention to other points.

Of these the chief and most prominent is, the *Progresssion of the Lunar Apogee;* partly from its intrinsic importance in furnishing to Newton's system one of the best and most satisfactory tests of its truth: and partly from its historical importance; for, an error committed in the first computations of its quantity made those who had adopted Newton's system to waver in their belief of its truth, and revived, for the same reason, the spirits of the drooping Cartesians.

This subject of the Progression of the Lunar Apogee has been already, in several places (see pp. 146. 157. 181.) adverted to; and, in fact, the substance of the source of the error and of the means of correcting the error, are already in the possession of the Student.

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* The integration of this equation, similar to that of the first would assign to δs an expression of this kind

$$\begin{aligned} &B\gamma \cdot \cos.(2v - 2mv \mp gv) \\ &\pm B' \cdot e\gamma \cdot \cos.(gv + cv) \\ &+ \&c. \end{aligned}$$

It is merely for his convenience, and for the purpose of a complete elucidation, that we collect its several parcels and arrange them in order.

A second subject of enquiry, connected with the preceding, but, like it, digressive, relates to the determination of the Progression of the Lunar Apogee from the consideration of one force alone acting in the direction of the radius. This, if the progression be rightly determined on the condition of *two* forces, one in the direction of the radius, the other tangential, may be thought a futile enquiry ; and, indeed, it deserves to be considered solely by reason of a sort of historical importance attached to it. Since some mathematicians, fancying themselves treading on the very footsteps of Newton, have sought for the quantity of the progression solely on the principles of the ninth Section.

These enquiries, if the main drift of the Treatise were merely the determination of the place of the disturbed planet, are not essential. And as, under any point of view, they partake somewhat of the nature of digressions, the Student will have the power of disregarding them as such, by passing over the next Chapter, which may be considered as separately assigned to them.

CHAP. XIII.

The Method of determining the Progression of the Apsides in the simplest Case of the Problem of the Three Bodies. Clairaut's Analogous Method for determining the Progression of the Lunar Apogee. His first Erroneous Result. Its Cause, and the Means of correcting it. Quantity of the Progression computed from the Condition of a Sole Disturbing Force acting in the Direction of the Radius Vector. Remarkable Result obtained by the first Integration of the Differential Equation. Dalembert's Method of Indeterminate Coefficients, for finding the Value of the Inverse of the Radius Vector, adopted by Thomas Simpson and Laplace.

THE simple instance of p. 109, &c., and which indeed is that which Clairaut (*Theorie de la Lune*, ed. 2. pp. 13, &c.) uses, will serve to illustrate that author's method of determining the Progression.

The general equation (see p. 109.) for determining u, in the Problem of the Three Bodies, is

$$\frac{d^2 u}{d v^2} + u - \frac{\mu}{h^2} - \Omega = 0,$$

if we make $\mu = 1$, and suppose the disturbing force to act solely in the direction of the radius vector and to be proportional to the inverse of its cube, we shall have (see p. 109.)

$$\Omega = \frac{m' u^3}{h^2 u^2} = \frac{m' u}{h^2},$$

and, accordingly, the differential equation will be

$$\frac{d^2 u}{d v^2} + u - \frac{1}{h^2} - \frac{m' u}{h^2} = 0;$$

This equation, if we make $m' = 0$ (which in fact is to suppose that there is no disturbing force), becomes the equation belonging to the elliptical system, and its integral determining u is of this form

$$u = \frac{1}{h^2}(1 + E \cos. v).$$

If this be the form for u in

$$\frac{d^2 u}{d v^2} + u - \frac{1}{h^2} = 0,$$

we may suppose a similar form

$$u = \frac{1}{p} + \frac{e}{p} \cos. c v,$$

to be the integral of

$$\frac{d^2 u}{d v^2} + \left(1 - \frac{m'}{h^2}\right) u - \frac{1}{h^2} = 0.$$

Clairaut, (see *Theorie de la Lune*, p. 13.) in order to verify the supposition, substitutes the assumed value of u in the differential equation; then, after the method described in pp. 99, &c. he integrates that equation, and compares the resulting value of u with the assumed; the former is

$$u = \frac{1}{h^2} + \frac{E}{h^2} \cos. v + \frac{m'}{p h^2} - \frac{m' e}{p (c^2 - 1)} \cos. c v$$
$$- \frac{m'}{p}\left(\frac{1}{h^2} - \frac{e}{c^2 - 1}\right) \cos. v,$$

which, compared with the latter, will give rise to three equations for determining the three arbitrary quantities p, e and c: these equations are

$$\frac{1}{h^2}\left(1 + \frac{m'}{p}\right) = \frac{1}{p},$$

$$\frac{E}{h^2} - \frac{m'}{p h^2} + \frac{m' e}{p (c^2 - 1)} = 0,$$

$$\frac{m'}{1 - c^2} = 1.$$

From these three equations will result three values for p, e and c, such as must make the assumed and deduced values of u perfectly to coincide.

The third is the important equation: from that we derive

$$c = \sqrt{(1 - m')},$$

and accordingly,

$$r = \frac{1}{u} = \frac{p}{1 + e \cos. \sqrt{(1 - m')} . v},$$

which is not the *approximate* but the *exact* equation for the radius vector of a body, acted on by a force compounded of two parts, one varying inversely as the square, the other inversely as the cube of the distance, and both, strictly speaking, centripetal, and not perturbative of the equal description of areas; although the latter, from analogy of the language used on these occasions, may be termed a disturbing force *.

The preceding equation (determining the value of r) although similar to, is not, in fact, the equation to an ellipse. But, after certain conventions, such as have been explained in pp. 119, &c. it will serve to represent the radius of a *moveable* ellipse; moveable in such a manner, that its axis-major revolves round the focus, as round a fixed point, with an angular velocity which is to that of the body revolving in the ellipse, as

$$1 - \sqrt{(1 - m')} \text{ is to } 1.$$

This angular motion of the axis is, in other words, the progression of the *apsides*, which are its extremities; or, in the case of the Moon revolving round the Earth, it is the progression of the Lunar Apogee; $1 - \sqrt{(1 - m')}$ expounding its quantity.

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* By the discoveries of Kepler the orbits of the planets appeared to be elliptical, and when afterwards they were found not to be strictly so, mathematicians were still inclined to view the ellipse as the *natural* curve, and consequently would term the peculiar law of force producing it, the *natural* law of force: other forces therefore which *disturbed* the elliptical form would be termed *disturbing* forces. 'Chaque planete decriroit *naturellement* une ellipse si elle n'etoit attireé que par le corps autour du quel elle tourne,' says Lalande, (*Astron.* tom. III. p. 596.) But, it is easy to see, these are merely the denominations of a conventional language.

In the preceding instance then, but in that alone, there is a perfect coincidence of the assumed and resulting values of u: there are three assumed arbitrary quantities, and three equations for determining them. If the disturbing force did not vary as the inverse cube of the distance, but as u^n, then (see pp. 123, &c.) the general differential equation will not assume the form

$$\frac{d^2 u}{d v^2} + N^2 u - \&c. = 0,$$

except the eccentricity of the orbit be very small; or, which amounts to the same thing, the value of u, such as (see pp. 111, &c.)

$$u = \alpha \,.\, \cos.\, N u + L,$$

will be only an approximate value. Moreover the value of N, on which the motion of the apsides depends, determined by the preceding method (pp. 109, &c.) will be only a near value: or (to make the phraseology approach to a similarity with that of Newton's) the body's place can be found, by the fiction of a moveable ellipse, only in orbits that are very nearly circular*.

But, as approximate solutions must be resorted to, when exact ones cannot be obtained, Clairaut supposed that he should obtain one of the former kind, when on the ground and principle of the exact solution (see pp. 102. 197. &c.) he compared the assumed value of u with its value resulting from the integration of the differential equation, in which, account had been made of both parts of the disturbing force; that is, of the tangential as well as of that which acts in the direction of the radius. The value resulting from integration was (see pp. 145. 156.) of this form †.

$$u = \frac{1}{a_{,}} - \frac{K}{2\,a_{,}} - \frac{3\,K e}{2\,a_{,}\,(c^2 - 1)} \left(1 + \frac{e^2}{2}\right) \cos.\, c\, v$$

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* Circulis finitimis, Newton, *Princ.* Sect.

† The following forms which may be easily made to coincide with Clairaut's are yet not exactly his. See *Theorie de la Lune,* pp. 23, &c.

$$+ \frac{Q}{a} \cos.\, (2\,v - 2\,m\,v)$$

$$+ \frac{Q'}{a}\, e \cos.\, (2\,v - 2\,m\,v - c\,v)$$

$$+ \&c.$$

$$+ \left(\frac{3\,K\,e}{2\,a_{,}\,(c^2 - 1)} - \frac{Q}{a} - \frac{Q'e}{a} - \&c.\right) \cos.\, v.$$

Now the assumed value of u is

$$u = \frac{1}{a}\,[1 + e^2 + e\,.\,(1 + e^2) \cos.\, c\,v],$$

which, compared with the former, gives three equations, (see p. 197.)

$$\frac{1}{a}\,(1 + e^2) = \frac{1}{a_{,}} - \frac{K}{2\,a_{,}},$$

$$\frac{1 + e^2}{a} = - \frac{3\,K}{2\,a_{,}\,(c^2 - 1)} \left(1 + \frac{e^2}{2}\right)$$

$$\text{and} \quad \frac{3\,K\,e}{2\,a_{,}\,(c^2 - 1)} - \frac{Q}{a} - \frac{Q'e}{a} - \&c. = 0,$$

from which (as before), the three arbitrary assumed quantities, a, c and e may be determined. The second equation ought, if the method were a right one, to determine c, and thence the progression of the Apogee.

Now if

$$e^2 = .0030107,$$

and (as it has been already shewn in p. 147.)

$$\frac{K\,a}{a_{,}} = m^2 = .005595,$$

there will result, very nearly,

$$c = .99581,$$

and

$$(1 - c)\, 360^0 = .00418 \times 360^0,$$

or the progression of the Apogee in a whole revolution, will equal

$1^{\circ}\ 29'\ 55''$ about half its real quantity, that is, half the quantity determined by observation [*].

This, however, it may be said, is only an approximate solution and necessarily incorrect; because, during the computation, several quantities dependent on the square and cube of the eccentricity, on the eccentricity of the Solar Orbit, on the inclination of its plane to that of the Moon's, &c. are neglected. But we have shewn in pp. 181, &c. that no retention of such quantities and account made of them, can ever correct the preceding error. The real correction consists in *repeating* the integration of the differential equation,

$$\frac{d^2 u}{d v^2} + u + \Pi = 0,$$

after the approximate value of Π has been formed, not by the assumed value of u, but by the value that results from the first integration (see pp. 185, &c.) [†].

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[*] Clairaut having computed, according to the preceding method, the value of c, drew this conclusion : 'Donc ou l'attraction Neutonienne ne donne point ce vrai mouvement ou la solution precedente n'est pas propre a la determiner.' Clairaut, *Theorie de la Lune*, ed. 2. p. 27. He had before said, in the Memoirs of the Academy, 'Apres avoir mis a ce calcul toute l'exactitude qu'il demandoit, j'ai eté bien etonné de trouver qu'il rendoit le mouvement de l'apogee au moins deux fois plus lent que celui qu'il a par les observations : c'est a dire que la periode de l'apogee qui suivroit de l'attraction reciproquement proportionelle aux quarrés des distances seroit d'environ 18 ans, au lieu d'un peu moins de 9 qu'elle est reellement', and 'Une resultat aussi contraire aux principes de M. Newton me porte *d'abord a abandonner entierement l'attraction*. *Mem. Acad.* 1745. pp. 336, 354.

[†] We have been very anxious to explain particularly and distinctly in what the real correction consists ; because it is frequently stated, (one author copying after another) that Clairaut committed his first error by neglecting to take account of certain terms, or by not pushing the approximation far enough : whereas, as it has been shewn, (pp. 181, &c.) it was not the neglecting of terms, but the *non* repetition of the process of approximation that was the cause of the error.

The real mode, however, of correcting the erroneous computation of the progression was by no means obvious. One proof of this is, that it eluded, for a time, Clairaut *, Dalembert and Euler, men of great sagacity and mathematical skill. For, as the Moon's Orbit was, very nearly, elliptical, the assumed elliptical value of the radius could not differ considerably from the resulting value. It seemed probable then that the comparison of the coefficients of like terms, which, in a simple hypothetical case, gave exact results, would, in this, give results nearly exact.

But, as it has been observed before, mere probabilities in such cases either determine nothing, or are fallacious. What is true of other terms is not so of that term which involves cos. $c\,v$. The peculiarity of its formation subjects it to a class of corrections from which the former are exempt.

These corrections have been given in pp. 185, &c. and Clairaut, in extricating himself from those embarrassments into which his first error had thrown him, shewed that he could correct, almost, completely, that error by taking account of the correction which the term

$$\frac{Q'}{a}\,e\,.\cos.\,(2\,v - 2\,m\,v - c\,v),$$

would introduce.

This undoubtedly is, when numerically expounded, the greatest correction which the coefficient of cos. $c\,v$ receives. It is (see p. 188.)

$$\frac{3\,K}{4\,a}\,Q'\left\{\begin{array}{c} 2 - 2\,m - c - 3 - \dfrac{4\,m}{2-2\,m-c} + \dfrac{1-(2-2m-c)^2}{1-m} \\ - \dfrac{8}{c} - \dfrac{8\,m}{c\,.\,(2 - 2\,m - c)} \end{array}\right\}$$

and this in numbers, supposing

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* Thomas Simpson, the ablest Analyst (if we regard the useful purposes of Analytical Science) that this Country can boast of, affirms in the Preface to his Tracts, that he himself, previously to any communication with M. Clairaut, found that the motion of the Apogee could be accounted for on the received Law of Gravitation.

$$Q' = .202,$$
$$m = .0748013,$$
$$* \ c = 1,$$

will be, very nearly,

$$- \frac{3\,K\,e}{2\,a_{,}}\,(1\,.\,1002).$$

If we use, therefore, this correction, we shall have, instead of the equation of p. 157. the following

$$1 + e^2 = - \frac{3\,K}{2\,a_{,}\,.\,(c^2 - 1)} \left(1 + \frac{e^2}{2} + 1.1002\right),$$

$$\text{whence, } 1 - c^2 = \frac{3\,K}{2}\,\frac{a}{a_{,}}\,.\,\left(1 - \frac{e^2}{2} + 1\,.\,1002\,.\,(1 - e^2)\right),$$

$$\text{(see p. 200.)} = \frac{3\,m^2}{2}\,.\,2\,.\,09518.$$

The † correction, therefore, arising from one additional term is a little more than equal the term to be corrected: and this *un-*

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* If c were really $= 1$, the progression of the Apogee would be nothing: but we are compelled, as in like cases, for the sake of approximation, to assume it at first of this value: for, c the quantity sought is involved in the expression of its value: we assume it, therefore, in the latter, of some determinate value, in order to escape from *a vicious circle.* The Science of Calculation abounds with such instances. If instead of $c = 1$, we had assumed it $= .99154801$, which we know from other sources to be its value, then, instead of the coefficient being $- \frac{3\,K}{2a}\,.\,e\,(1.1002)$, it would have been $- \frac{3\,Ke}{2\,a}\,(1.1009)$, so that the faulty assumption of $c = 1$, in the involved expression for its value, introduces no greater error than $- \frac{3\,Ke}{2\,a}\,.007$.

† Clairaut's correction (*Theorie de la Lune*, ed. 2. pp. 27, &c.) for the term $\mathcal{Q}'\,e\,.\,\cos.\,(2\,v - 2\,m\,v - c\,v)$, is *nearly* the same as what we have deduced; not exactly, since in finding the variation of $\frac{m'}{2h^2}\,.\,\frac{u'^3}{u^3}$ cos. $2\,\omega$, he neglects to take account of the variation of cos. $2\,\omega$. And the authority of Clairaut has served to entail this error on some subsequent authors.

expected fact, if we may so call it, at once dispelled those doubts which Clairaut entertained of the truth of Newton's Law of Attraction.

To be perfectly assured of the truth of that Law as established by this instance, it is necessary to add the corrections due to the other additional terms, of which the most considerable are

$$\frac{Q''}{a} e \,.\, \cos.\, (2v - 2mv + cv), \text{ and } \frac{Q}{a} \cos.\, (2v - 2mv).$$

The correction due to this latter is (see pp. 185, 186, &c.),

$$\frac{3K}{a} Q e \left[[4 \,.\, (1-m)^2 - 1] \,.\, \left(\frac{1+m}{2-2m-c} + \frac{1-m}{2-2m+c} \right) \right] \cos.\, cv,$$

and this, if we suppose

$$Q = .007092,$$
$$m = .0748013,$$

is nearly equal to $\frac{3m^2}{2} e \times .05419$.

The correction due to $\frac{Q''}{a} e \,.\, \cos.\, (2v - 2mv + cv)$ is (see pp. 188, &c.)

$$\frac{3K}{4a} Q'' e \left\{ \begin{array}{c} 2 - 2m + c - \left(3 + \dfrac{4m}{2-2m+c} \right) + \dfrac{1-(2-2m+c)^2}{1-m} \\ + \dfrac{8}{c} + \dfrac{8m}{c \,.\, (2-2m+c)} \end{array} \right\}$$

which, if Q'' be supposed equal to .003729, nearly equals $\frac{3K}{2a} e \times .0004765$.

If therefore we find an equation for c, after having applied the three corrections which arise from the terms whose arguments are

$$2v - 2mv - cv, \; 2v - 2mv, \text{ and, } 2v - 2mv + cv,$$

we shall have

$$\frac{e}{a}(1 + e^2) \,.\, (1 - c^2) =$$

$$\frac{3Ke}{2a_{,}} \cdot \left(1 + \frac{e^2}{2} + 1 \cdot 1002 - .05419 + .0004765\right);$$

$$\text{whence,} \quad 1 - c^2 = .017132, \text{ nearly,}$$

$$\text{and } c = .9914, \text{ nearly;}$$

$\therefore (1 - c)\, 360^0$, or the progression in a whole revolution, is equal to $.0085 \times 360^0$, that is, to $3^0\ 3'\ 18''$, which result is considerably within a minute of the true result, or of that which is determined by observations *.

This *accounting for* the Progression of the Lunar Apogee, or the exact computation of its quantity by means of the Law of Gravity, was the first great addition made to the system of Newton after the publication of the *Principia:* it afforded, although not a complete confirmation of the truth of the Law of Gravitation, yet a strong argument in its favor: for, as the perfect quiescence of the apsides is one of the most simple of the results of the Law of Gravitation, in the system of two bodies, so their *Progression* is one of the most remarkable effects of the perturbation of that system: when, as in the Lunar Theory, the interference of a third attracting body deranges the laws of elliptical motion.

It has been just said that the agreement of the computed and observed progression was the first great addition made to Newton's system after the publication of the *Principia.* For, it must be remarked that the *quantity* of the progression is not computed in any part of that extraordinary Work. It was, in no other scientific treatise, attempted to be computed by Newton, nor, on just and intelligible principles, by any of his contemporaries. Clairaut, first computed it, and by aid of what is called the Modern Analysis †.

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* There are other corrections to the coefficients of cos. cv, than those we have taken notice of; for instance, the corrections due to the eccentricity of the Solar Orbit and to the variation of s have been neglected.

† Amongst the many instances of results derived from the analytical method, and, so it would seem, above the power of the geometrical, this is one. It appears even difficult, on the grounds of the latter method, to shew in a general way, that the progression of the apsides must take place in consequence of the agency of a disturbing force.

If the progression of the Lunar Apogee be truly computed from the consideration of two disturbing forces, it is rather in the nature of an idle discussion to enquire whether its right quantity will result when the condition of the tangential force is suppressed. Still the question is not altogether unworthy of notice. It has some historical interest; for, Newton computes the motion of the apsides solely when the disturbing force acts in the direction of the radius vector; and some writers believe it to have been his opinion, that the progression of the Lunar Apogee could be rightly determined on that condition*, and, the most celebrated of the Commentators of the *Principia* have endeavoured so to compute it. But besides this, the enquiry is interesting from a curious circumstance of computation attached to it.

If the perpendicular force T be made equal nothing, then (see pp. 169, &c.), we shall have this reduced value of Π,

$$\Pi = -\frac{1}{h^2} + \frac{3\gamma^2}{4h^2} + \frac{3\gamma^2}{4h^2}\cos. 2gv + \frac{m'u'^3}{2h^2u^3} + \frac{3m'u'^3}{2h^2u^3}\cos. 2\omega.$$

Now, of these terms the only one that involves cos. cv is $\frac{m'}{2h^2} \cdot \frac{u'^3}{u^3}$, and (see pp. 153. 157.) from its evolution there arises

$$-\frac{3K}{2a}\left(1 + \frac{e^2}{2}\right) . e \cos. cv.$$

If, therefore, after the first approximation and integration, we compare, as Clairaut originally did, the corresponding terms of the assumed and resulting values of u, we shall have, amongst other equations, this

$$\frac{e}{a}(1 + e^2) = \frac{3Ke}{2a(1 - c^2)}\left(1 + \frac{e^2}{2}\right);$$

which (see p. 157.) is precisely the equation that resulted from the comparison of the coefficients of cos. cv, when both parts of the disturbing force, the one in the direction of the radius vector, the other tangential, were amongst the conditions of the problem.

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* 'Though there are some who have, both before and since, undertaken to give the true quantity of that motion, from such principles, only, as are laid down in the ninth Section of the first Book of the *Principia'*, *&c.* Simpson's *Tracts*, Preface.

Here then we have the curious circumstance to which (see pp. 195. 206.) we alluded. If Clairaut's first method of finding the Progression of the Lunar Apogee had been right, it would inevitably have followed, that the progression was altogether independent of the tangential force: or, in other words, that it depended solely on that part of the disturbing force which acts in the direction of the radius vector.

But, as it has been abundantly shewn, Clairaut's first method of determining the progression was wrong. A correction was omitted nearly equal to the quantity to be corrected. Now the same cause of error which affected Clairaut's computation must also affect the previous computation of the coefficient of cos. $c\,v$, and a similar correction must be applied.

Now the sole correction of the coefficient of cos. $c\,v$, when the tangential force T is equal nothing, must be derived (see Table, p. 175.) from the term $\frac{m'}{2\,h^2}\cdot\frac{u'^3}{u'^3}$ cos. $2\,\omega$: and that correction will be had by writing respectively for q, $2-2\,m-c$, and $2-2\,m+c$, and by designating by A and B what Q becomes in these cases: the first part of the correction will then be

$$-\frac{3\,K}{2\,h^2}\cdot A\cdot\left(\frac{3}{2}+\frac{2\,m}{2-2\,m-c}\right),$$

the second,

$$-\frac{3\,K}{2\,h^2}B\cdot\left(\frac{3}{2}+\frac{2\,m}{2-2\,m+c}\right),$$

where A and B represent the coefficients of

e . cos. $(2\,v-2\,m\,v-c\,v)$, and e . cos. $(2\,v-2\,m\,v+c\,v)$, when $T=0$.

If we revert to pp. 204, &c. we shall find on excluding from the formulæ those parts which are derived from T, or which depend on T, that

$$A=\frac{3\,K}{4\,a\,[1-(2-2m-c)^2]}\,(3+4\,m)=.052,$$

$$B = -\frac{3K}{4a[(2-2m+c)^2-1]}(3-4m) = -.0015.$$

$$\text{Hence, } A\left(\frac{3}{2} + \frac{2m}{2-2m-c}\right) = .08636,$$

$$\text{and } B\left(\frac{3}{2} + \frac{2m}{2-2m+c}\right) = -.00232.$$

Since $T = 0$, there will be no correction due to the term $Q \cdot \cos. (2v - 2mv)$: the corrected coefficient of $\cos. cv$, therefore, will be nearly

$$-\frac{3Ke}{2a}(1.08858),$$

and accordingly the equation for determining c will be

$$1 - c^2 = \frac{3m^2}{2} 1.08858 \times .996698;$$

$$\text{whence } c = .9954,$$

$$\text{and } (1-c)v = .0045v;$$

if therefore we substitute 360° instead of v, the progression in a revolution will be about 1° 37′ 12″, a quantity somewhat exceeding the half of the true result.

Hence it appears that, on assigning to the parts of the disturbing force their just values, the progression of the apogee depends, for nearly half its value, on the tangential force, and for the remainder on that part of the disturbing force which acts in the direction of its radius; a conclusion widely different from that (see p. 147.) which the result from the first approximation afforded.

The method of finding the progression of the Lunar Apogee by the comparison of the coefficients of the cosines of like *arguments* is capable of great accuracy. It originated, as we have often said, from Clairaut: but it has not been constantly adopted. M. Laplace, for instance, who seldom treads in the exact steps of his predecessors, has found the quantity of the progression, and its *secular equation* by a different method (see *Mec. Celeste*, 2de Partie. Liv. VII. pp. 212, &c.)

This Chapter, as we premised, is somewhat beside the main course of investigation; and we will augment still more its digressive nature by briefly commenting on the method, by which Thomas Simpson and Laplace have obtained the coefficients of the general equation,

$$\frac{d^2 u}{d v^2} + u + \Pi = 0.$$

The method which these two mathematicians use may be characterised as that of *indeterminate coefficients*. It was first suggested by Dalembert (*Theorie de la Lune*, pp. 107, &c.) who, however, does not adopt it, but employs for his practical solution, one of approximation and integration similar to that which has been described (pp. 137, &c.). The method of indeterminate coefficients Dalembert recommends as a good one, care being taken previously to ascertain the *form* of the series to be determined *; by which he means that the multiple arcs, or arguments (such as $2v-2mv$, $2v-2mv-cv$, &c.) according to the cosines of which the series is to be arranged, must be previously determined. Now this caution is observed both by Thomas Simpson and Laplace. The former in his *Miscellaneous Tracts*, p. 148. first approximately integrates the differential equation in order to discover † the arguments or arcs, the cosines of which would be involved in the terms of the series for the inverse of the radius vector (u), and then assumes a series for u, the terms of which are the products of the cosines of the deduced arcs and of certain arbitrary quantities, such as B, C, &c.

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[*] 'Cette maniere d'appliquer la methode des indeterminés a la solution d'une probleme dont il s'agit, est sans comparaison la plus courte et la plus facile de toutes, puis qu' elle ne demande ni integration ni aucun addresse de calcul.' p. 107. Again, 'Cette methode exige quelques precautions, pour ainsi dire, preliminaires; sçavoir, de prouver que la forme qu'on suppose a l'equation est en effet la seule qu' elle doive avoir. Or, j'ai cru qu'il etoit plus court de chercher directement cette forme en integrant rigoureusement et absolutement l'equation proposée,' &c. p. 109, &c.

[†] 'But, since the former operation is made, more with a view to discover the form of the series, than to be regarded for its exactness, I shall have no further reference thereto, but proceed to determine the several quantities e, B, C, &c. *de novo*, by a method somewhat different from that used above.' p. 148.

Now, if we revert to pp. 184, &c. it will appear that the necessity of correcting the coefficients of the terms of the series arose from Π having been deduced from the elliptical and imperfect value of u. The corrections successively arise on restoring to u its deficient terms: they will, therefore, be of necessity superseded if the component parts of Π be, in the first instance, deduced not from the elliptical value of u, but from that series which, with regard to its form at least, rightly represents its value. What will require to be done more than was done in pp. 185, &c. is the determination of the assumed arbitrary or indeterminate coefficients: and for this purpose there will be an equal number of equations.

The erroneous determination of c arose, as we have seen, from the component parts of Π having been deduced from the imperfect and elliptical value of u. That error, therefore, must necessarily be avoided by this method of Simpson, which, in the first instance, is founded on what may be viewed as a complete representation of the value of u; c, therefore, is determined with as much exactness as the method of approximation (for after all we are still thrown back on such methods) will admit of. And this Simpson states to be one of the advantages of his method *.

Laplace in his *Mecanique Celeste*, (tom. III. pp. 191, &c.) although, in the main, he follows Dalembert's suggested method, yet follows it not so closely as Simpson has done. He first, on the assumption of the *elliptical* value of u, deduces the values of the coefficients of the terms of the differential equation, and expresses them by means of the quantities m, e, e', c, &c. Observing then the forms of those terms that would constitute the increment to the elliptical value of u arising from the disturbing force, Laplace assumes (δu representing the above-mentioned increment),

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* 'It not only determines the motion of the apogee in the same manner, but utterly excludes, at the same time, all terms of that dangerous *species* (if I may so express myself) that have hitherto embarrassed the greatest mathematicians, and that would, after a great number of revolutions, entirely change the figure of the orbit.' Simpson's *Tracts*, Preface.

$$\begin{aligned} a\,\delta u = {} & Q' \,.\, \cos.\,(2v - 2mv) \\ & + Q'' e \,.\, \cos.\,(2v - 2mv - cv) \\ & + Q''' e \,.\, \cos.\,(2v - 2mv + cv) \\ & + \&c. \end{aligned}$$*

The next step in Laplace's process is to correct the value of Π, previously obtained on the ground of the elliptical value of *u*, by supposing *u* to vary, and its variation (δu) to have that form which has been just assigned to it.

The last operation of Laplace's is to substitute in the differential equation which resulted from the previous operations (the coefficients of the terms being compounded of *m*, *e*, *e′*, *c′*, &c. and of *Q′*, *Q″*, *Q‴* &c.) for *u*, this value

$$u = \frac{1}{a}\left(1 + e^2 + \frac{\gamma^2}{4} + e\cos.\,cv - \frac{\gamma^2}{4}\cos.\,2gv\right) + \delta u\,;$$

thence will result an identical equation such as

$$A + B\cos.\,cv + C\cos.\,(2v - 2mv) + De\,.\,\cos.\,(2v - 2mv - cv) + \&c.$$

in which, *A*, *B*, *C*, &c. will be (to use a general term) *functions* of *e*, *m*, *c*, and of *Q′*, *Q″*, *Q‴*, &c. and, for the determining of these latter quantities, (for they being known the variation of *u* arising from the disturbing force will be known) there will be these equations,

$$\begin{aligned} A &= 0, \\ B &= 0, \\ C &= 0, \\ &\&c. \end{aligned}$$

(see Laplace, *Mec. Cel.* Partie 2de. Liv. VII. pp. 215, &c.)

From this brief account, besides for the reasons stated in pp. 209, 210, &c. it will appear that no error, nor any semblance of error, in the determination of the progression of the apogee, similar to that which occurred in Clairaut's first Essays, can take place in this method.

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* The additional terms due to the disturbing force have the same form in the differential as in the integral equation that assigns the value of *u*. For instance, if $P\cos.\,pz$ be a term in the former, then $\frac{P}{p^2 - 1}\,.\,\cos.\,pz$ is the corresponding term in the latter.

Although the method of indeterminate coefficients is a sure and excellent one, yet it has not been adopted in these pages. Instead of it, we have employed another less scientific, perhaps, but more simple and obvious, more in unison with preceding methods and better suited to the purpose and plan of the Treatise. Nor are these latter advantages counterbalanced by any incorrectness. For by means of the Table and formulæ (see p. 175.) the method is capable of receiving a series of successive corrections.

We will now resume the main course of investigation, and proceed to the solution of the second equation (see p. 95.); thence we shall have t in terms of v, and consequently, the mean anomaly in terms of the true; but the solution depends (see p. 95.) on the tangential force T and on u. The value of this quantity u, therefore, requires to be known previously to the determination of the time. It is not, therefore, without reason that the equation [*b*] of p. 95. claims precedence of consideration: and the deduction of the value of u is, perhaps, of not less importance for collateral purposes than for the obvious and direct one of determining the parallax.

CHAP. XIV.

Expression for the Time: first, when the Body revolving in a Circular Orbit is disturbed by the Action of a very distant Body. The Mean Longitude expressed in Terms of the True: the True thence expressed in Terms of the Mean by the Reversion of Series. The Introduction of Inequalities in the Mean Motion by the Disturbing Force: the Elliptic Inequality, the Variation: the greatest Value of the latter in an Orbit nearly Circular. Expression for the Differential of the Time in an Elliptical Orbit, the Disturbing Body revolving also in an Orbit of the same kind. The Expression integrated, and the Mean Longitude expressed in Terms of the True. Expression in this Case, of the Coefficient or greatest Value of the Variation. The Secular Equation of the Mean Motion, explanatory of the Acceleration of that Motion. Digression concerning the Properties and Uses of the Formula of Reversion. By means of that Formula the True Longitude expressed in Terms of the Mean: the Terms expound Inequalities: the greatest denominated the Variation, the Evection, the Annual Equation, the Reduction: Causes of their Magnitude. Lunar Tables, in what manner, improved by Theory.

THE general equation, (p. 95.)

$$dt = \frac{dv}{hu^2\sqrt{\left(1 + \frac{2}{h^2}\int \frac{T\,dv}{u^3}\right)}},$$

which expresses the differential of the time in a *disturbed* orbit, is reduced, when T the tangential disturbing force $= 0$, to (see p. 96.)

$$dt = \frac{dv}{hu^2},$$

and this latter (see p. 14.) is the analytical expression of Kepler's Law of the Equable description of Areas.

By means of these two expressions, the deviation from Kepler's Law may be computed: but, it is to be observed, their sole difference does not consist entirely in the last term of the denominator of the former which involves T, since u, in fact, is different in the two expressions.

The value of u in the latter expression, if the plane of the body's orbit be supposed inclined to another plane, and γ denote the tangent of inclination, is

$$u = \frac{1}{a}\left(1 + e^2 + \frac{\gamma^2}{4} + e \,.\, \cos.\, c\,v - \frac{\gamma^2}{4}\cos.\, 2\,g\,v,\right)$$

and of h we have this value

$$h = \sqrt{[a\,.(1 - e^2 - \gamma^2)]}.$$

If the condition of the smallness of the eccentricity be such, that terms involving the cube and higher powers of the eccentricity may be rejected, then, by first expanding $\frac{1}{u^2}$ and next by integrating

$$d\,t = \frac{d\,v}{h\,u^2},$$

there will result

$$n\,t = v - 2\,e\,.\sin.\, c\,v + \frac{3}{4}\,e^2\,.\sin.\, 2\,c\,v + \frac{\gamma^2}{4}\sin.\, 2\,g\,v,$$

from which expression we may rescind the last term, if the inclination of the planes of the orbits be very minute.

The preceding expression gives us, within certain limits of exactness, the mean longitude of a body describing an ellipse in terms of the true: but, in order to compare the observed with the true place of a planet, or, in order to construct Astronomical Tables which will assign for any epoch the planet's true place, we require a formula assigning v in terms of $n\,t$. Such a formula we may deduce from Lagrange's Theorem, (see *Trig*. Appendix, p. 213.), and by an use of it, similar to that which has been already made of it in pp. 31, 32, when W was deduced in terms of $n\,t$. If we apply then such theorem, make $c = 1$, and reject terms that involve the cube of the eccentricities or products of the square of the eccentricity and tangent of inclination, we have

$$v = nt + 2e \sin. nt + \frac{5e^2}{4} \sin. 2nt + \frac{\gamma^2}{4} \sin. 2gnt.$$

If the approximation were continued, and more terms were taken account of, then the additional terms would involve in their coefficients, e^3, e^4, $e\gamma^2$, &c. and have for their arguments,

$$3nt, \ 4nt, \ 2gnt \pm nt, \text{ \&c.}$$

But, as it has been observed, the above values of nt and v belong to the elliptical system; in the *disturbed* system they will be changed for two causes; an alteration in the value of u, and in the denominator of the fraction expressing dt (see p. 213.).

The values of u and of $\int \frac{T dv}{u^3}$ have already been assigned in the preceding pages. We might then, by one effort, obtain a general solution and assign nt in terms of v. But, as it is the drift of the present Treatise to conduct the Student, through the more simple, to the investigation of the complex cases, we shall first deduce an expression for the mean motion in terms of the true, when the body revolving in a circular orbit is disturbed by the action of a very remote body.

Let δu designate the alteration in u, or its variation produced by the disturbing force, then

$$dt = \frac{dv}{h.(u+\delta u)^2}\left(1 + \frac{2}{h^2}\int \frac{T dv}{u^3}\right)^{-\frac{1}{2}}$$

$$= \frac{dv}{hu^2}\left(1 - \frac{2\delta u}{u}\right)\left(1 - \frac{1}{h^2}\int \frac{T dv}{u^3}\right), \text{ nearly,}$$

$$= \frac{dv}{hu^2}\left(1 - \frac{2\delta u}{u} - \frac{1}{h^2}\int \frac{T dv}{u^3}\right), \text{ nearly.}$$

Now, in a circular orbit, (see pp. 133, &c.)

$$\delta u = -E.\cos. v + L.\cos. (2v - 2mv),$$

and (see p. 130.)

$$\frac{1}{h^2}\int \frac{T dv}{u^3} = \frac{3Ka}{4.a_{,}(1-m)} \cos. (2v - 2mv);$$

but, see pp. 133. 136,

$$L = \frac{3K}{2a_{,}}\left(\frac{2-m}{1-m}\right) \cdot \frac{1}{(2-2m)^2-1}\text{, nearly,}$$

$$\text{and } E = \frac{K}{2a_{,}} - \frac{3K}{2a_{,} \cdot (2-2m)^2-1} \cdot \frac{2-m}{1-m},$$

if therefore we substitute in the preceding expression for dt, δu expressed by means of the above quantities, and the value of $\frac{1}{h^2}\int \frac{Tdv}{u^3}$, there will result after integration

$$t = \frac{a^2}{\sqrt{a_{,}}}\, v$$

$$- \frac{3Ka^3}{2(1-m)^2\sqrt{a_{,}^3}}\left(\frac{2-m}{(2-2m)^2-1} + \frac{1}{4}\right) \sin.(2v-2mv)$$

$$+ \frac{Ka^3}{\sqrt{a_{,}^3}}\left(1 - \frac{3}{(2-2m)^2-1} \cdot \frac{2-m}{1-m}\right) \sin. v.$$

Substitute $\frac{1}{n}$ instead of $\frac{a^2}{\sqrt{a_{,}}}$, and

$$nt = v - q \cdot \sin.(2v-2mv) + p \cdot \sin. v,$$

q and p standing for the coefficients of sin. $(2v - 2mv)$ and sin. v in the reduced expression.

This case being intended, almost entirely, for illustration, we have assumed only that increment (δu) of u which results from the first approximation and integration. In consequence of this assumption, nt has been increased by only two terms: instead of being equal to v, which it would be in a circular orbit, nt now equals $v - q.\sin.(2v - 2mv) + p.\sin. v$: if the middle term were rescinded, the equation,

$$nt = v + p \cdot \sin. v,$$

would express the relation between the mean and true anomalies in an ellipse of very small eccentricity (see pp. 31, 32.). But, as it is plain from l. 8, 9, the coefficients p, q, are, with regard to magnitude, of the *same order*. The disturbing force, then, (under such conditions as have been explained in l. 15, 16.) affects at once the mean circular motion with two *inequalities;* one elliptic, of which the argument is v, or, nearly, nt; the other (see *Astronomy*, pp. 326, &c.) technically called the *Variation*, and of which the argument is $2v - 2mv$, or, nearly, $2nt - 2mnt$.

From the first approximate solution then of the equation [*a*], (see p. 95.) as well as from that of equation [*b*], (see pp. 95, 134.) it follows that a circular orbit is not changed into an elliptical by the influence of the disturbing force.

$nt - mnt$ expresses the mean angular distance of the revolving and disturbing body. In the Lunar Theory, accordingly, it denotes the mean angular distance of the Sun and Moon, and it is frequently thus symbolically expressed,

☽ − ☉;

to the sine of double this quantity, that is, to sin. 2 (☽ − ☉), the Lunar *Variation* (see *Astron.* pp. 326, &c.) is proportional; or, more correctly, the principal term of the *Variation* involves sin. 2 (☽ − ☉).

The Variation and other equations (see *Astron.* Chap. XXXIV.) are applied as corrections to the mean anomaly for the purpose of deducing the true. In order, therefore, to deduce analytically these equations or corrections, we must, by means of the formula of reversion, express v in terms of nt and of other quantities.

In the simple case we have taken (that of the perturbation of a body revolving in a circular orbit),

$$nt = v + p \, . \, \sin. v - q \, . \, \sin. (2v - 2mv),$$

and, if we examine the formula (see *Trig.* Appendix, pp. 213, &c.) by which v is to be expressed in terms of nt, &c., it will immediately appear that, to every *argument* in the original expression, there must be, at the least, a corresponding argument in the reversed expression. This is effected by the second term (yX) in the formula of reversion which necessarily introduces terms such as $P \, . \, \sin. nt$, $Q \, . \, \sin. (2nt - 2mnt)$. But the third term of the formula $\left(\frac{y^2}{2}, \frac{d(X^2)}{n\,dt}\right)$ will introduce additional terms depending on new arguments, involving, however, smaller coefficients than the preceding terms. These new arguments will be

$$2nt, \quad 4nt - 4mnt,$$
$$nt - 2mnt, \quad 3nt - 2mnt.$$

If we stop at the third term of the formula *,

$$v = nt - p \, . \sin. nt + q \, . \sin. (2nt - 2mnt)$$

$$+ \frac{p^2}{2} \, . \sin. 2nt + \frac{q^2}{2} (2 - 2m) \, . \sin. (4nt - 4mnt)$$

$$+ \frac{pq}{2} (1 - 2m) \, . \sin. (nt - 2mnt) - \frac{pq}{2} \sin. (3nt - 2mnt),$$

and the values of p, q, being those which are assigned at p. 216. it follows, that the four last terms are much smaller than the three preceding.

The argument of the elliptic inequality (see *Astronomy*, Chapters XVIII. and XXXIV.) is nt; and, of the Variation, $2(\text{☽} - \odot)$ is the argument; and, were the preceding value of v an exact one, $-p \, . \sin. nt$, and $q \sin. 2(\text{☽} - \odot)$ would be the principal terms of those inequalities. But, as it has been already observed (p. 217.) they are not the sole terms: for, it is usual to consider the terms that involve the sines of multiples of the argument of the principal term of an inequality as belonging to, and partly expounding it: so that, in the present instance,

$$-p \, . \sin. nt + \frac{p^2}{2} \sin. 2nt,$$

would expound the elliptic inequality, and,

$$q \, . \sin. 2(\text{☽} - \odot) + q^2 \, . (1 - m) \sin. 4(\text{☽} - \odot),$$

the *Variation.*

The *Variation* (which is Newton's *Acceleration of Areas*, see *Prin.* Prop. xxvi. Lib. III.) originates, as we see by the preceding case, from the disturbing force; and, almost entirely, from the tangential disturbing force. The disturbing force in the direction of the radius has some influence in producing it, inasmuch as the variation (δu) of u, in part, arises from such force, (see pp. 124, &c. &c.)

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* In order to obtain, what this formula enables us to do, v in terms of nt, &c. Clairaut, in his *Theorie de la Lune*, ed. 2. pp. 59, 60, 61. propounded, but without their demonstrations, three Lemmas; and, for the same end Lalande has investigated a formula in his *Calcul des Inegalités de Venus par l'attraction de la Terre,* see *Acad. des Sciences,* 1760. pp. 326, &c.

An inequality, such as the Variation (although that term has been in a way appropriated to the Lunar Theory) must affect the motion of any body, revolving round another, and disturbed by a third: Venus, therefore, Jupiter, a Satellite of Jupiter, must, in their motions, be subject to such an inequality; and, indeed, to several of the same sort: to as many as there are disturbing bodies. Venus, then, taking that planet for our instance, is subject to *variations* from the Earth, Mars, Jupiter, Saturn, and the Georgium Sidus; of unequal magnitude indeed, and some so small as not to be worth considering.

In the Lunar Theory the coefficient q of the principal term of the Variation is considerable: it equals (see p. 216.) $a_{,}$ being supposed $= a$,

$$\frac{3K}{2 \,.\, (1-m)^2}\left(\frac{2-m}{(2-2m)^2-1}+\frac{1}{4}\right)$$

Now, (see p. 132.) $K = .005595$,

and, since $m = .0748013$;

$q = .01021$, nearly; or, in degrees, &c. $= 35' \; 4''$,

and, accordingly, the principal term of the Variation is

$$35' \; 4'' \,.\, \sin.\, 2\,(☽ - ☉).$$

This coefficient of the principal term of the Variation is much nearer the true value *, than one would have been led to expect from the preceding imperfect value of $n\,t$: so imperfect, indeed, that of all the noted Lunar inequalities the *Variation* is the only

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* Mayer in his *Theoria Lunæ*, p. 52. represents the Variation by

$$\begin{aligned} &- \; 1' \; 55'' \,.\, \sin.\, D, \\ &+ \; 35 \,.\, 47 \,.\, \sin.\, 2D, \\ &+ \; 2 \,.\, \sin.\, 3D, \\ &+ \; 14 \,.\, \sin.\, 4D, \end{aligned}$$

D being the same as ☽ − ☉.

one deducible from it: for it is plain, from the nature of the formula of reversion, if

$$nt = v + p\,.\,\sin.\,v - q\,.\,\sin.\,(2v - 2mv),$$

that the arcs involved in the terms expounding the value of v (or the arguments of the equations correcting its value) can only be those which are formed by the addition and subtraction of

$$nt,\ \ 2nt - 2mnt,\ \ 2nt,\ \ 4nt - 4mnt, \text{ \&c.}$$

which combination, although it will produce an indefinite number of arcs, will never produce the arguments belonging to the *Evection* and *Annual Equation* (see *Astron.* Chap. XXXIV.)

These inequalities, like the Variation and others, affect the motion of the Moon revolving round the Earth and disturbed by the Sun: that case, however, is most inadequately represented by the preceding instance: for, to go no farther, the orbits were there supposed devoid of eccentricity and inclination. It is not, however, the mere omission of these conditions that is the sole cause why the true anomaly v (see p. 218.) is so inadequately represented.

The main reason is the deduction of nt from that first imperfect value of δu which results from the first approximation and integration. If these latter processes be, as they ought to be, repeated, then such a value of δu will result from them, as substituted in the expression for $\int dt$ (see p. 215.) will supply to nt (and consequently, see p. 217, after reversion, to v) those terms that analytically expound the *equations* that are used in correcting the Moon's mean longitude, (see *Astron.* Chap. XXXIV.)

In deducing $u + \delta u$, and $\int \frac{T\,dv}{u^3}$, u (see pp. 130, 131.) was supposed constant $\left(= \frac{1}{a}\right)$: consequently those terms, which involve the eccentricity and on which several of the Lunar inequalities (the Evection for instance,) depend, could not result from the first processes of approximation and integration: but they would have resulted had, in the first instance, an elliptical value been given to u: they must result then when the process of integration is repeated, or when a second value of δu and a second value of $\int \frac{T\,dv}{u^3}$ is deduced, by substituting in their expressions,

$$\frac{1}{a} - E\,.\cos.\,v + L\,.\cos.\,(2\,v - 2\,m\,v),$$

which is the value of u resulting from the first integration: for, the two first terms belong to an ellipse: and, therefore, the substitution of such value of u must give rise to, at least, as many and as various terms as the substitution of u's elliptical value would.

The effect of the disturbing force (as we have seen in p. 217.) does not change the circular into an elliptical, but, as we may consider it by reason of the value of u (see p. 130.), into a disturbed elliptical orbit: there will be then, from this mode of considering the subject, as many different terms in the resulting value of u, and $\int \frac{T'\,dv}{u^3}$, and consequently as many different terms in the value of $n\,t$, as if the orbit, before perturbation, had been supposed elliptical.

But, although the disturbing force will render the circular orbit eccentric, it can never render its plane inclined to that of the disturbing body, if the planes be originally coincident. No repetition, then, of process can ever introduce into the values of u and v terms depending on the inclination; such must originate from the first substituted value of u, when it contains a term dependent on the inclination.

This value, in an elliptical orbit, is

$$u = \frac{1}{h^2\,(1 + \gamma^2)} \left(1 + \frac{\gamma^2}{4} + e\,.\cos.\,cv - \frac{\gamma^2}{4}\cos.\,2gv\right),$$

if we substitute it and deduce δu by the methods described in pp. 159, &c. we shall have *

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* Laplace in expressing the value of $a\,\delta u$ by a series of terms, uses coefficients such as $A_1^{(1)}$, $A_2^{(0)}$, &c. in which the figure at the top denotes

$$\begin{aligned} a\,\delta u = {} & A^{(0)} . \cos. (2v - 2mv) \\ & + A^{(1)} . e . \cos. (2v - 2mv - cv) \\ & + A^{(2)} . e . \cos. (2v - 2mv + cv) \\ & + A^{(3)} e' . \cos. (2v - 2mv - c'mv) \\ & + A^{(4)} e' . \cos. (2v - 2mv + c'mv) \\ & + \&c. \\ & + A^{(5)} \gamma^2 \cos. 2gv \\ & + A^{(6)} e' . \cos. c'mv, \end{aligned}$$

and see pp. 155. 168.

$$1 - \frac{1}{h^2}\int\frac{Tdv}{u^3} = 1 - \frac{3m^2}{2}\left\{\begin{aligned} & \frac{1+2e^2}{2-2m} . \cos. (2v - 2mv) \\ & - \frac{2+2m}{2-2m-c} . e \cos. (2v - 2mv - cv) \\ & - \frac{2-2m}{2-2m+c} . e \cos. (2v - 2mv + cv) \\ {}^{*} & + \frac{7}{2.(2-3m)} . e' \cos. (2v - 2mv - c'mv) \\ & - \frac{1}{2.(2-m)} . e' \cos. (2v - 2mv + c'mv) \\ & + \&c. \end{aligned}\right\}$$

therefore, since (see p. 215.)

$$\begin{aligned} dt = {} & \frac{dv}{hu^2}\left(1 - \frac{1}{h^2}\int\frac{Tdv}{u^3}\right) - \frac{dv}{hu^2} . \frac{2\,\delta u}{u} \\ & + \frac{dv}{hu^2} . \frac{2\,\delta u}{u} . \frac{1}{h^2}\int\frac{Tdv}{u^3}, \end{aligned}$$

(where the last term is very small), we shall have, by substituting for $1 - \frac{1}{h^2}\int\frac{Tdv}{u^3}$ and δu their preceding values, and then integrating, a more correct value of t; in deducing which, as it is plain, not solely the conditions of the eccentricities and inclination are taken account of, but the correction to the value of δu arising

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denotes the order of arrangement, and the figure at the bottom the degree of minuteness: which, in a Work like his, is a convenient method (see *Mec. Celeste*, Liv. VII. p. 200.)

* $2-3m$, $2-m$ are written instead of $2-2m-c'm$, $2-2m+c'm$, to which they are nearly equal.

from a repetition of the process by which it is found (see pp. 184, &c.)

The expanded value of $d\,t$ will consist of $d\,v$ multiplied into a constant coefficient, and, besides, of $d\,v$ multiplied into a series of terms involving the cosines of arcs, such as $c\,v$, $2\,v - 2\,m\,v$, &c. consequently of a non-*periodical* and of a *periodical* part. With regard to the former; the constant coefficient of $d\,v$, if we reject terms that involve the square of the disturbing force, will arise (as it is plain from the inspection of the terms composing $d\,t$) from the expansion of $\frac{1}{h\,u^2}$.

$$\text{Now, } \frac{1}{h\,u^2} = h^3\left(1 + \frac{3}{2}e^2 + \frac{3}{2}\gamma^2\right)$$

$$- 2\,h^3\,e\left(1 + \frac{3}{2}e^2 + \frac{5}{4}\gamma^2\right)\cos.\,c\,v$$

$$+ \frac{3\,h^3\,e^2}{2}\cos.\,2\,c\,v,$$

$$+ \frac{h^3\,\gamma^2}{2}\cos.\,2\,g\,v$$

$$- \&c.$$

the time therefore of a revolution (which is independent of the terms that involve the cosines or sines of arcs) is equal to

$$\int d\,v\,.\,h^3\,.\left(1 + \frac{3}{2}e^2 + \frac{3}{2}\gamma^2\right);$$

but this same time (see p. 147.) $= \int \frac{d\,v}{h\,u^2} = \frac{a^2}{\sqrt{a_{,}}}$;

$$\therefore\ h^3 = \frac{a^2}{\sqrt{a_{,}}\left(1 + \frac{3}{2}e^2 + \frac{3}{2}\gamma^2\right)}$$

$$= \frac{a^2}{\sqrt{a_{,}}}\,.\left(1 - \frac{3}{2}e^2 - \frac{3}{2}\gamma^2\right), \text{ nearly.}$$

Hence, the whole value of $\frac{d\,v}{h\,u^2}$ is thus to be expressed,

$$\frac{d\,v}{h\,u^2} = \frac{a^2}{\sqrt{a_{,}}}\,.\,d\,v\left\{\begin{array}{l} 1 - 2\,e\,.\left(1 - \frac{\gamma^2}{4}\right)\cos.\,c\,v + \frac{3}{2}e^2\,.\cos.\,2\,c\,v \\ \qquad + \frac{\gamma^2}{2}\cos.\,2\,g\,v \\ \qquad + \&c. \end{array}\right\}$$

this (see p. 222.) is to be multiplied into $1 - \frac{1}{h^2}\int\frac{T\,d\,v}{u^3}$. The quantity to be multiplied into $\delta\,u$, is

$$\frac{2\,d\,v}{h\,u^3} = \frac{a^3\,d\,v}{\sqrt{a_{,}}}\left\{\begin{array}{l} 2 + e^2 - \frac{\gamma^2}{2} - 6\,e\,.\cos.\,c\,v \\ + \frac{3}{2}\,\gamma^2\,.\cos.\,2\,g\,v + \&c. \end{array}\right\}$$

If we make $\frac{a^2}{\sqrt{a_{,}}} = \frac{1}{n}$, perform the necessary multiplications, and then integrate the expression for $d\,t$, we shall have

$$\begin{aligned} n\,t = v &+ B\,e\,.\sin.\,c'v + B'\,e^2\,.\sin.\,2\,c\,v \\ &+ C\,.\sin.\,(2\,v - 2\,m\,v) \\ &+ C'e\,.\sin.\,(2\,v - 2\,m\,v - c\,v) + C''\,e\,.\sin.\,(2\,v - 2\,m\,v + c\,v) \\ &+ C_{,}e'.\sin.\,(2\,v - 2mv - c'\,m\,v) + C_{,,}\,e'.\sin.\,2v - 2mv + c'mv) \\ &+ D\,.\,\gamma^2\,.\sin.\,2\,g\,v + \&c. \\ &+ E\,e'\,.\sin.\,c'\,m\,v + \&c. \\ &+ \&c. \end{aligned}$$

and the values of B, B', &c. C, C', &c. may easily be deduced from the preceding formulæ, (see pp. 222, &c.)

For instance, if we omit the last term (see p. 222.) of the expression for $d\,t$, then

$$B = -\frac{2\,.\left(1 - \frac{\gamma^2}{4}\right)}{c},$$

and if, for the sake of greater exactness, we retain it,

$$B = -\frac{2\,.\left(1 - \frac{\gamma^2}{4}\right) + \frac{3\,m^2\,A^{(1)}}{4\,.(1 - m)}}{c},$$

the argument $(c\,v)$ of the additional term being introduced by the multiplication of the first term of $\frac{1}{h^2}\int\frac{T\,d\,v}{u^3}$ (see p. 222.) with the second term of $a\,\delta\,u$, since

$$\cos.\,(2\,v - 2\,m\,v)\,.\cos.\,(2\,v - 2\,m\,v - c\,v) = \tfrac{1}{2}\cos.\,c\,v + \&c.$$

C is the coefficient of the term on which (see p. 217.) a principal Lunar inequality, namely, the *Variation*, depends. Now it is plain, that the multiplication of the first term of $\frac{1}{h^2}\int\frac{T\,d\,v}{u^3}$ with the constant terms of $\frac{1}{h\,u^2}$, and of the second and third terms of $\frac{1}{h^2}\int\frac{T\,d\,v}{u^3}$ with the second term of $\frac{1}{h\,u^2}$ (that which involves cos. $c\,v$) will produce terms dependent on the angle or argument $2\,v - 2\,m\,v$. Terms dependent on the same angle will also be produced by the multiplication of the constant part of $\frac{2}{h\,u^3}$ (see pp. 222. 224.) with the first term of $\delta\,u$, and by the multiplication of the fourth term of $\frac{2}{h\,u^3}$ with the second and third terms of $\delta\,u$. If these operations be carried into effect, we shall have

$$(2 - 2m)\,C = -\frac{3\,m^2.(1+2\,e^2)}{4\,.(1-m)}$$

$$-\,3\,m^2\,e^2\left(\frac{1+m}{2-2m-c}+\frac{1-m}{2-2m+c}\right)$$

$$-\,2A^{(0)}\left(1+\frac{e^2}{2}-\frac{\gamma^2}{4}\right)+3e^2.(A^{(1)}+A^{(2)}),$$

and, similarly, the coefficients of the other terms may be deduced.

In the preceding process for deducing the value of $d\,t$, the constant coefficient of $d\,v$, on neglecting the squares of $\delta\,u$ and $\int\frac{T\,d\,v}{u^3}$, is expressed by $\frac{a^2}{\sqrt{a_{,}}}$; in this coefficient, a is the semi-axis of the Lunar Orbit, and $a_{,}$ (a quantity to be determined by calculation) the semi-axis, such as would belong to the Moon's orbit, were it not disturbed by the Sun's action. Now, (see pp. 180. 182.)

$$\frac{1}{a}=\frac{1}{a_{,}}\left(1+e^2+\frac{\gamma^2}{4}\right)-\frac{K}{2\,a_{,}}\left(1+e^2+\frac{\gamma^2}{4}+\frac{3\,e'^2}{2}\right);$$

consequently,

$$a^2 = a_{\prime}^2 \left(1 - e^2 - \frac{\gamma^2}{4}\right) + \frac{3K}{2} a_{\prime}^2 . e'^2, \text{ very nearly,}$$

$\frac{a^2}{\sqrt{a_{\prime}}} dv$, therefore, must contain a term such as $\frac{3K}{2} a_{\prime}^{\frac{3}{2}} e'^2 . dv$. But, as it will be shewn in a following Chapter, e' the eccentricity of the Solar Orbit is, from the disturbing forces of the planets, subject to a secular Variation: the term, therefore, just obtained must be separated, in the integration, from the other part of the coefficient of v. Making, therefore, (which is nearly true) $\frac{1}{n} = a_{\prime}^{\frac{3}{2}}$, we have, independently of the terms that involve the sines of arcs,

$$nt = v + \frac{3K}{2} \int e'^2 . dv + \text{correction},$$

$$= v + \frac{3m^2}{2} \int [(e'^2 - E'^2) . dv],$$

m^2 being (see p. 132.) nearly equal to K.

In reversing the preceding expression of nt in order to obtain v, the two first terms in its resulting value will be (see pp. 217, 218.)

$$nt - \frac{3m^2}{2} \int [(e'^2 - E'^2) n\,dt],$$

the mean motion, therefore, which depends on these two first terms, will not be constant, but will be subject to a *secular Variation*, of which the second term is the exponent. And this is the mathematical explanation, on Newton's Principle of Gravitation, of that phenomenon which is called the *Acceleration of the Moon's Mean Motion* (see *Astronomy*, p. 312.). The explanation was first given by Laplace, and it is to be reckoned amongst the most excellent of the results which the analytical method * of treating *Physical Astronomy* has afforded.

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* By this is meant the *method* which originated with Clairaut, Dalembert and Thomas Simpson, and which has been so successfully followed by Lagrange and Laplace.

In the first instance, the value of $n\,t$, in consequence of the disturbing force, was increased by two terms, on one of which the inequality called the Variation depended; and, it was observed at p. 219. that, although the conditions of the case differed, in so many respects, from the real conditions in the Lunar Theory, yet the *coefficient* of the variation (in other words the greatest value of the equation), was nearly of its just value. The value, therefore, of that coefficient must depend, in a slight degree only, on the elliptical form of the orbit *, and on the disturbing force which acts in the direction of the radius. The actual difference of the two coefficients may easily be computed from the preceding expression, (see p. 225.) for, since $A^{(0)}$, (rejecting the terms that involve $m^2 e^2$, $m^2 \gamma^2$, &c.) is equal

$$\frac{3\,m^2}{2} \cdot \frac{2-m}{(1-m)\,4\,(1-m)^2-1}\text{, we have (see p. 225.)}$$

$$C = -\frac{3\,m^2}{8\,.\,(1-m)^2} - \frac{3\,m^2\,.\,(2-m)}{2\,.\,(1-m)^2\,[4\,.\,(1-m)^2-1]}$$

$$-\frac{3\,m^2 e^2}{4\,(1-m)^2} - \frac{3\,m^2 e^2}{2\,(1-m)} \cdot \left(\frac{1+m}{2-2m-c} + \frac{1-m}{2-2m+c}\right)$$

$$-\frac{A^0}{1-m}\left(\frac{e^2}{2} - \frac{\gamma^2}{4}\right) + \frac{3\,e^2}{2(1-m)}\;(A^{(1)} + A^{(2)}),$$

Now here the two first terms

$$= -\frac{3\,m^2}{2\,(1-m)^2} \cdot \left(\frac{2-m}{4\,.\,(1-m)^2-1} + \frac{1}{4}\right);$$

but (see p. 132.) $m^2 = K$ nearly; these two first terms, then, are the value of q † (see p. 219.), or of the coefficient of the Variation in the first simple case: the remaining terms of C, therefore, express the defect of its just value.

The quantity C does not strictly represent the coefficient of the Variation: in the first case it did, since then the coefficient of $2\,v - 2\,m\,v$ and of $2\,n\,t - 2\,n\,m\,t$ were the same: but in the present case, if

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* See Newton, Prop. 26. 29. Book III.

† With the sign changed: for, if $+A$ sin. $a\,v$ be a term in the series for $n t$, $-\,A$ sin. $a\,n\,t$ will be the corresponding term in the series for v.

$$nt = v + Be\,.\,\text{sin.}\,cv + C\,\text{sin.}\,(2v - 2mv),$$
$$+\, C'e\,.\,\text{sin.}\,(2v - 2mv - cv) + \text{\&c.}$$

v, by the process of Reversion, will not be justly represented by

$$nt - Be\,\text{sin.}\,cnt - C\,.\,\text{sin.}\,(2nt - 2mnt);$$

for, the combination of sin. cnt with sin. $(2nt - 2mnt - cnt)$ which takes place in X^2* will produce one term involving cos. $(2nt - 2mnt)$ in $\frac{d(X)^2}{ndt}$; there will, therefore, be a term involving sin. $(2nt - 2mnt)$, and the coefficient of this will, as it is evident, serve to augment the value of C.

In the first case, the true longitude v was expressed by means of three terms; one the mean motion; the second expounding the first or elliptic inequality; the third expounding the Variation; and, in other language, we might say, in such a case, that, in order to find the true longitude of a body, it is necessary to correct the mean longitude by two equations, one *the Equation of the Centre* (see *Astronomy*, p. 322.) the other the Variation, and of which

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[*] If nt contain no other terms than what are stated in l. 1, then (see *Trig.* pp. 213, &c.)

$$y = -\,Be$$

$$X = \text{sin.}\,cnt + \frac{C}{Be}\,\text{sin.}\,(2nt - 2mnt) + \frac{C'}{B}\,\text{sin.}\,(2nt - 2mnt - cnt).$$

Now, X^2 with other terms will contain this,

$$2\,\text{sin.}\,cnt.\,\frac{C'}{B}\,.\,\text{sin.}\,(2nt - 2mnt - cnt) = \frac{C'}{B}\,\text{cos.}\,(2nt - 2mnt - 2cnt)$$
$$-\,\frac{C'}{B}\,\text{cos,}\,(2nt - 2mnt);$$

$\therefore\ \frac{y^2}{2}\,.\,\frac{d(X)^2}{ndt}$ will contain $\frac{B^2e^2}{2}\,.\,\frac{C'}{B}\,.\,(2 - 2m)\,.\,\text{sin.}\,(2nt - 2mnt)$,

which is the only term in $\frac{y^2}{2}\,.\,\frac{d(X)^2}{ndt}$ that involves sin. $(2nt - 2mnt)$.

Hence,

$$v = nt - Bc\,.\,\text{sin.}\,cnt - [C - BC'e^2\,.\,(1 - m)]\,\text{sin.}\,(2nt - 2mnt) - \text{\&c.};$$

and the next term $\frac{y^3}{6}\,.\,\frac{d^2(X)^3}{ndt}$ will produce another small addition to C.

the argument is twice the angular distance of the disturbed and disturbing body. In the last case we have taken, nt, being expressed by a series of many terms that are functions of v, v, by the formula of Reversion, will consist, (see p. 217.) at the least, of as many with similar arguments: it will, in fact, contain more with new arguments: so that the form of v will be thus expressed,

$$
\begin{aligned}
v = nt - \frac{3m^2}{2} & \int (e'^2 - E'^2) \,.\, ndt \\
& + Pe \,.\, \text{sin.}\, cnt + P'e^2 \,\text{sin.}\, 2cnt \\
& + Q \,.\, (\text{sin.}\, 2nt - 2nmt) \\
& + Q'e\,(\text{sin.}\, 2nt - 2nmt - cnt) + \&\text{c.} \\
& + R\gamma^2 \,.\, \text{sin.}\, 2gnt + \&\text{c.} \\
& + Se' \,.\, \text{sin.}\, c'nmt + \&\text{c.} \\
& + \&\text{c.}
\end{aligned}
$$

in which P, P', Q, &c. will differ from the values of B, B', C, &c.; the coefficients of the largest terms but little, the coefficients of the small terms more, according, however, to no rule or formula of difference.

The additional terms will be introduced by the third, fourth, &c. terms of the formula of Reversion: for instance, $\frac{y^2}{2} \,.\, \frac{d\,(X^2)}{ndt}$ will introduce terms involving the sines of

$$
\begin{gathered}
2nt - 2nmt \mp 2gnt, \quad 4nt - 4nmt, \\
2nt - 2nmt \mp cnt \mp c'nmt, \ \&\text{c.}
\end{gathered}
$$

The forms of the several arguments of the terms by which nt is expressed, will, it is evident, be all produced in the expression for v, by means of the second term (yX) of the formula of reversion: they will again be produced by the fourth, sixth, &c. terms of the formula: for suppose

$$P \,\text{sin.}\, pv + Q \,.\, \text{sin.}\, qv,$$

to be two terms of the expression for nt; then,

$$- P \,\text{sin.}\, pnt - Q \,\text{sin.}\, qnt,$$

are produced by the second term: but the fourth term is $\frac{y^3}{2.3}\cdot\frac{d^2.(X)^3}{(n\,dt)^2}$.

Now, X^3 will produce, besides other terms,

$$P^3\sin.^3 pnt + Q^3\sin.^3 qnt$$
$$+\,3P^2Q\sin.^2 pnt\sin. qnt + 3Q^2P\sin.^2 qnt\,.\,\sin. pnt,$$

or, see *Trig*. p. 54,

$$\frac{3}{4}P^3\sin. pnt + \frac{3}{4}Q^3\sin. qnt - \&c.$$
$$+\,\frac{3}{2}P^2Q\sin. qnt + \frac{3}{2}Q^2P\sin. pnt - \&c.$$

$\therefore\ \frac{1}{2.3}\cdot\frac{d^2.(X)^3}{(n\,dt)^2}$ will produce, in the expression for v,

$$-\left(\frac{P^3p^2}{8}+\frac{PQ^2p^2}{4}\right)\sin. pnt,$$
$$-\left(\frac{Q^3q^2}{8}+\frac{QP^2q^2}{4}\right)\sin. qnt.$$

Hence, (since $\frac{d.(X)^2}{n\,dt}$ will produce no term involving $\sin. pnt$, or $\sin. qnt$), if we go no farther than the fifth term of the formula, we shall have, supposing

$$nt = v + P\sin. pv + Q\sin. qv,$$
$$v = nt - P\left(1-\frac{P^2p^2}{8}-\frac{Q^2p^2}{4}\right)\sin. pnt$$
$$-\,Q\left(1-\frac{Q^2q^2}{8}-\frac{P^2q^2}{4}\right)\sin. qnt,$$

and if we add to nt an additional term $R\sin. rv$, then

$$v = nt - P\left(1-\frac{P^2p^2}{8}-\frac{Q^2p^2}{4}-\frac{R^2p^2}{4}\right)\sin. pnt$$
$$-\,Q\left(1-\frac{Q^2q^2}{8}-\frac{P^2q^2}{4}-\frac{R^2q^2}{4}\right)\sin. qnt,$$
$$* - R\left(1-\frac{R^2r^2}{8}-\frac{P^2r^2}{4}-\frac{Q^2r^2}{4}\right)\sin. rnt,$$

---

* This agrees with Clairaut's formula, p. 61. *Theorie de la Lune*, p. 61.

in which expression, we have the corrected coefficients of sin. $p n t$, sin. $q n t$, sin. $r n t$, on the condition that, in the process of Reversion, we do not go beyond the term $\frac{y^3}{2.3} \cdot \frac{d^2.(X^3)}{(n d t)^2}$. The succeeding term of the formula will not produce a term involving sin $p n t$: but the next following will: for, to go no farther than the first term of X^5, which is $P^5 . \sin.^5 p n t$: this (see *Trig.* p. 54.) contains a term $P^5 \frac{10}{16} \sin. p n t$; and consequently, $\frac{d^4.(X^5)}{(n d t)^4}$ will produce a term $P^5 \frac{10}{16} . p^4 . \sin. p n t$. Other terms of X^5 will also produce terms involving sin. $p n t$; but, in the case we are treating of (the expression of the true anomaly in terms of the mean) the coefficients of these terms are so minute as not to be worth taking account of.

If the process of Reversion be not carried beyond the fourth term of the formula, then, as we have seen,

$$- P\left(1 - \frac{P^2 p^2}{8} - \frac{Q^2 p^2}{4} - \frac{R^2 p^2}{4}\right),$$

is the coefficient of sin. $p n t$. It is also its *complete* coefficient, if no combination of the other arcs ($q n t$, $r n t$, &c.) such as $q n t \pm r n t$, $2 q n t \pm r n t$, &c. equals $p n t$, or can produce it. Hence, if in p. 224. $n t$ were correctly expressed by the series of terms there given, since no combination of the other arcs could possibly produce $2 g v$, the coefficient ($R \gamma^2$) of the corresponding sine (sin. $2 g n t$), would be similar to the above coefficient of sin. $p n t$.

The same may be said of the term $E e' . \sin. c' m v$: for there is no combination of the other arcs that can produce $c' m v$; but, as we have seen in pp. 224, 225. the case is quite different with several of the other terms. The arc, for instance, which is the argument of the Variation, can be formed by the combination of $c v$ with $2 v - 2 m v - c v$, and $2 v - 2 m v + c v$, and also by the combination of $c' m v$ with $2 v - 2 m v - c' m v$, and $2 v - 2 m v + c' m v$. In the process of Reversion, therefore, Q, besides that value

$$\left[\text{of the form } - P\left(1 - \frac{P^2 p^2}{8} - \frac{Q^2 p^2}{4} - \frac{R^2 p^2}{4}\right)\right]$$

which it would have, did the combination of no other arcs form $2v - 2mv$, must have certain additional terms, involving the coefficients of cv, $2v - 2mv - cv$, $2v - 2mv + cv$, &c: Let the series for nt and v be those that are stated in p. 224, then

$$Q = -C\left(1 - \frac{C^2 . (2-2m)^2}{8} - \frac{B^2 e^2 . (2-2m)^2}{4} - \&c.\right)$$
$$+ (BC' - BC'') . (1 - m)\, e^2$$
$$+ (EC_{,} - EC_{,,}) . (1 - m) e'^2,$$

in which, as we have stated, the first line on the right-hand side of the equation, is of that form which is common to the coefficient of every term in the reversed expression: the second and third lines are peculiar to the coefficients of those terms alone, the arcs or *arguments* of which can be formed by the combination (the addition or subtraction) of other arguments.

These observations on the process of *Reversion* and on the method of expressing the true longitude in terms of the mean, are here inserted, not because the subject is very abstruse, but because it is rarely and imperfectly treated of. The subject is indeed of an analytical nature, and related to the present investigations in no other degree than by belonging to one of the Sciences that are auxiliary to Physical Astronomy.

We will now resume the investigation of the main subject of this Chapter, which is, (if any other be in such researches,) very interesting, and which directly bears on some of the most essential points of Newton's System.

We have already seen in the simple case, in which the body was supposed to revolve originally in a circular orbit, that the disturbing force added two new terms to the original equation,

$$nt = v,$$

or caused nt to be thus expressed,

$$nt = v + p . \sin. v - q \sin. (2v - 2mv),$$

in which p and q are coefficients dependent on the disturbing force.

The Reversion of this expression produced (see p. 218.), (neglecting the squares, &c. of p, q) this equation,

$$v = nt - p \sin. nt + q . \sin. (2nt - 2mnt);$$

or, v the true longitude no longer equal to the mean, became *unequal* to it by two *inequalities*, one expounded by $-p . \sin. nt$, the other by $q \sin. (2nt - 2mnt)$: the first called the Elliptic Inequality *, the second the *Variation.*

These and like *inequalities* are said, in technical language, to be corrected by their corresponding *equations ;* for instance, corresponding to the two inequalities just mentioned are the *Equation of the Centre* and *the Variation* (which is an *equation*) of which, in the case we are now speaking of, the terms $p \sin. nt$, $q . \sin. 2 (☽ - ☉)$ are the mathematical exponents.

According then to the conventional language just described, the body's true longitude is to be found by applying to the mean longitude the *Equation of the Centre,* and the equation called the *Variation:* the *argument* of the former equation being the Moon's anomaly, of the latter, twice the angular distance of the Sun and Moon.

In the more complex case (see p. 229.), (that in which most of the real conditions that obtain in nature are introduced) v is expressed by a great variety of terms, each of which, like the term $p . \sin. 2 (☽ - ☉)$, may be said to denote an *inequality* or to expound an *equation:* the *coefficient* depending on the disturbing force, and the *argument* on the *Configuration* of the Sun, Moon and Earth.

The body's true longitude then in this case is to be found (on the supposition that the series of terms in p. 229. truly represents v) by correcting the mean longitude $(n t)$ first by a *secular equation* $\left(\frac{3 m^2}{2} \int (e'^2 - E'^2)\, n dt\right)$, and then by other *periodical equations* of which the coefficients are $P e$, $P' e^2$, Q, $Q' e$, &c. and the respective *arguments*

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* In this instance $p \sin. nt$ which arises entirely from the disturbing force, is not strictly the *Elliptic Inequality:* since that term is appropriated to the deviation from equable motion which ensues from the merely elliptical motion.

cnt, $2cnt$, $2nt - 2nmt$, $2nt - 2nmt - cnt$, &c.;

or, making A to represent the Moon's mean anomaly, the arguments will be

$$A,\ 2A,\ 2(☽ - ☉),\ 2(☽ - ☉) - A,\ \&c.$$

But, it is plain, if we look merely to the mathematical or symbolical exhibition of terms, their number is infinite. Except, then, several should be eminent above the rest for their magnitude, or, to state the matter more correctly, except after certain terms, all succeeding terms should be so minute as not to be worth considering, the true longitude could not be computed either directly from the preceding series, or by its aid.

The fact is that all terms, saving about thirty, may, from their minuteness, be rejected; and, if we examine the series we shall perceive the causes of the minuteness; since, past a certain term, the coefficients involve the cubes of the eccentricities and inclination and their products.

The terms then to be retained and to expound *equations*, are considerably the greatest of the series, and their magnitude, or, rather, that of their coefficients depends, if we regard the differential equation subsisting between dv and dt, on two causes; the magnitude of that modification of the disturbing force which produces the inequality; and, the duration of its agency. Thus, if

$$dv = ndt + \&c. + P.\cos.pnt \times ndt;$$

then, dv will, in a given element of time (dt), be the greater, the greater P is; P being a function of the disturbing force and other quantities such as the eccentricity, &c. The difference between the true longitude and the mean will continue to increase whilst $P\cos.pnt$ remains of the same sign. The whole excess, therefore, of the true above the mean longitude, will depend (P being given) on the length of time that $P\cos.pnt$ continues of the same sign; and, that must depend on p: the smaller p, the greater will be the period of $\cos pnt$ passing from a given positive value to an equal negative one: the larger p, the quicker will be the transition from the positive to the negative values of $\cos.pnt$.

This is to view the subject on certain intelligible grounds of cause and effect: but, by the mathematical process, we may arrive, and in a more summary way, at the same result: for if

$$dv = n\,dt + \&c. + P\cos.pnt \times n\,dt,$$

$$v = nt + \&c. + \frac{P.\sin.pnt}{p},$$

and v will be the greater, the greater P is and the smaller p.

The Variation then, which is one of the principal Lunar Equations, must derive its magnitude from that of the modification of the disturbing force producing it: since, $2nt - 2nmt$ being its argument, the divisor $2 - 2m$ introduced by integration is greater than 1. Or, in the other mode of considering the matter, we may say that the intensity of the disturbing force must be considerable, since the whole period of its action does not exceed fifteen days *.

If we apply the mathematical mode of estimating the magnitudes of the terms to the expression for dt, we shall easily see what are the kind of terms that give rise to considerable equations. Now the expression is

$$dt = \frac{dv}{hu^2} - \frac{dv}{h^3u^2}\int\frac{T\,dv}{u^3} - \frac{dv}{hu^2}\cdot\frac{2\,\delta u}{u}.$$

---

* The exact period is 14,765294, or half a synodic period. During this, the variation passes through all its degrees of magnitude, positive as well as negative: the time, therefore, during which the Variation either continually augments, or continually diminishes the longitude, can be only one-fourth of a synodic period. In order to deduce this result, we may observe that sin. z passes through all its positive and negative values, whilst z passes from 0 to a value $=360°$: now, $2nt-2mnt=0$, when $t=0$: and, in order to find that value of t which makes $2nt-2mnt=360°$, we have, n denoting the Moon's motion in a given time (a day for instance),

$$360° : n :: ☽\text{'s period} : 1;$$

$$\therefore\ 2nt - 2mnt = n\,.\ ☽\text{'s period, and}$$

$$t = \frac{☽\text{'s period}}{2.(1-m)} = \tfrac{1}{2}(☽\text{'s synodic period}).$$

Here the last term involves δu, the variation of u arising from the disturbing force; and this variation is obtained by integrating the equation,

$$\frac{d^2 u}{d v^2} + u + \Pi = 0,$$

and if $P \cos. p v$ (see p. 101.) should be a term in Π, then $\frac{P . \cos. p v}{p^2 - 1}$ would be the corresponding term introduced by integration into the value of u. If p then should be nearly equal 1, there would be introduced, on that account, into the value of δu, and consequently into that of $d t$, a term with a large coefficient. Now, if we examine the expression for δu, we shall find the third term somewhat under the above predicament: the argument of that term is

$$(2 - 2m - c) v.$$

In the Lunar Theory $m = .0748013$,

$$c = .99154801,$$

and consequently,

$$2 - 2m - c = .85885,$$

the term therefore in δu that involves the argument $(2 - 2m - c)v$ must have received by the integration a small quantity as its divisor; when the integral of $d t$ is taken, the second integration will introduce the divisor $2 - 2m - c$, which will not, however, much affect the value of the coefficient.

The term we are now speaking of expounds, in the Lunar Theory, the equation which is called the *Evection*, and which, together with the Variation (see *Astron.* Chap. XXXIV.) was discovered long before the rise of *Physical Astronomy*.

The second integration, we have just seen, introduces, if the term in $d t$ be $A . \cos. (2v - 2mv - cv)$, a divisor $2 - 2m - c$. That quantity differing little from 1 does not much alter the resulting coefficient $\left(\frac{A}{2 - 2m - c}\right)$: but, it is plain, if in $P \cos. p v$ a term of $d t$, p should be very small, that the coefficient of the resulting term would be considerably increased by integration: now the last term $A^{(6)}e' \,.\cos. c'mv$ in the value of $a\,\delta u$ (see p. 222.) is nearly in this predicament; for, since

$$c' = .99999077965,$$

$$\text{and } m = .0748013,$$

$$c'm = .07480059;$$

consequently the term $A^{(6)}e' \,.\cos. c'mv$, which is very small in the expression for dt, will become of some magnitude $\left(= \frac{A^{(6)}e' \sin. c'mv}{c'm}\right)$ in the integrated expression.

The term that we have been just considering expounds, in the Lunar Theory, the *Annual Equation*, (see *Astronomy*, Chap. XXXIV.) discovered, as the *Variation* and *Evection* were, long before the rise of *Physical Astronomy*.

Thus, the simple consideration of the divisors introduced by integration has helped us to the *mathematical* explanation of the magnitudes of two of the principal Lunar Equations. The magnitude of the *Variation* is derived from that of the modification of the disturbing force producing it. Almost the whole of the tangential disturbing force is so expended. (see p. 218.)

The cause however of the magnitude of the *annual equation* is easily to be discerned without the aid of the *mathematical* explanation. It is owing to the *duration* of that modification of the disturbing force which produces it: which duration is half the period of a Solar Anomalistic Year *. The magnitude of the *Evection* arises not from any acceleration in the direction of the

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* For reasons already stated in the note of p. 235. the whole period of the annual equation is to be determined from this equation,

$$c'mnt = 360^\circ = n\,.\,\text{☽'s period};$$

$$\therefore\; t = \frac{\text{☽'s period}}{c'm} = \frac{\oplus\text{'s period}}{c'} = \text{Solar Anomalistic year:}$$

the annual equation, therefore, increases continually, or diminishes continually, during only half an Anomalistic year, the longitude.

tangent, but from alterations produced in the ellipse by that portion of the disturbing force which acts in the direction of the radius. There is an inequality due solely to the eccentricity of the orbit. The equation of the centre becomes greater by the ellipse becoming more eccentric. If therefore the disturbing force introduces any periodical change in the eccentricity, there will be a corresponding change in the equation of the centre; something superadded to, or taken away from that *equation*, which is independent of all disturbing force, and which is solely due to the original ellipse: and this periodical augmentation and diminution is a new *Equation*: it is, in the case we are considering, the *Evection*, arising from the disturbing force in the direction of the radius altering, what some mathematicians have been pleased to call, the Natural Centripetal Law. We see by this, why we must look for the magnitude of the Evection in the integration of the equation [*b*] of p. 95. which determines the radius vector. The merely mathematical cause of the magnitude of the term expounding the Evection consists, as we have seen in p. 236., in its receiving a small divisor $(2 - 2m - c)^2 - 1$.

The principle that has enabled us to assign the cause of the preceding Lunar equations will serve to guide us in our search of others. The Annual Equation, as we have seen, although originating from a small modification of the disturbing force, yet becomes considerable by the accumulation of its effects. A slight modification of the disturbing force corresponds, in the differential equation, to a small coefficient: and the accumulation of effects depends on the period of the inequality. We must, therefore, in reducing the differential equation by the rejection of small terms, be careful to examine those terms that expound inequalities of long periods. The lengths of the periods may always be determined on the principles laid down in pp. 235. 237.

The three equations, the *Variation*, the *Evection* and the *Annual Equation*, have claims to particular consideration from their historical celebrity. They have received, unlike the other inequalities that depend on the disturbing force, a technical denomination. If they be viewed solely with regard to their magnitude, they will be found to belong to the *second order* of inequalities: the elliptic inequality being reckoned of the first; of a superior order it certainly is: for, its coefficient is 6° 18′, whereas the coefficients of the Variation, Evection, and Annual Equation, are 35′ 46″, 1° 20′ 29″, and 11′ 11″ respectively. But these inequalities do not exclusively occupy the second order: for, of the same order, (always determining the order by the magnitude of the coefficient) is the *Second Elliptic Inequality*, as it may be called, of which, A being the Moon's Anomaly, $2A$ is the argument, and the coefficient (the greatest value of the equation) 12′. There is also another inequality of the same order denominated the *Reduction:* which, like the two elliptic inequalities, is almost entirely independent of the disturbing force. The two latter principally depend on, or are derived from, the elliptic form of the orbit; whilst the *reduction* is owing to the inclination of the plane of the orbit to that of the ecliptic.

There are then five inequalities of the second order, and one of the first; and the true longitude expressed by the six terms that are their exponents, would be (reckoning the anomalies from perigee,)

$$v = nt + (6^{\circ}\ 17'\ 54')\ \text{sin.}\ A,$$

Second Inequality	$+\ 12' \,.\, \text{sin.}\ 2A,$
Variation	$+\ 35' \,.\, 46'' \,.\, \text{sin.}\ 2(☽ - ⊙),$
Evection	$+\ 1^{\circ} \,.\, 20' \,.\, 29''\ \text{sin.}\ [2(☽ - ⊙) - A],$
Annual Equation	$+\ 11' \,.\, 11'' \,.\, \text{sin.}\ a,$
Reduction	$-\ \ 7' \,.\, 31'' \,.\, \text{sin. 2 dist.}\ ☽\ \text{from}\ ☊,$

a representing the Sun's Anomaly.

These are the principal terms of the series expressing the value of v. If we continue the examination of the terms, we may select and arrange, into a third class or order, fifteen other terms expounding inequalities: of which the arguments would be

$$2(☽ - ⊙) + A, \quad 2(☽ - ⊙) - a,$$
$$2(☽ - ⊙) - A + a, \quad A - a, \quad 3A,$$
$$☽ - ⊙, \text{\&c. \&c.}$$

After this third class we may carry the approximation still farther and form a fourth; and, as it has been observed (p. 234.)

there can be no end, in a merely mathematical view of the subject, of the terms composing the value of v; or, to use a different phraseology, an infinite number of equations result from theory to be applied as corrections to the mean longitude for the finding of the true. Of these it is sufficient to retain twenty-eight, the others being rejected; the sole rule and guide for rejection being their ascertained or computed minuteness *.

The *coefficients* of the terms (which are the greatest values of those terms) are constant. But the Moon, and Earth, the places of their apsides, and nodes, the inclination of the planes of their orbits; or, what technically is so called, the *Configuration* of the Earth, Moon and Sun, continually varying, the arguments which depend on such configuration, must continually vary: they must change from day to day. The Moon's place, therefore, if assigned on the principles of *Physical Astronomy*, would require every day the computation of nearly thirty terms (such as the terms of p. 229.). This would be very laborious. But in this, as in like cases, the labour, although it must always remain considerable, is lessened by the construction of *Lunar Tables* †.

Lunar Tables are not constructed solely by means of *Physical Astronomy*; nor, in fact, do they essentially require its aid. Three of the principal equations were determined, their coefficients as well as their arguments, long before Newton's discoveries. As they were determined so might the other equations, although the requisite labour of computation and observation would have been very great. Theory brings us sooner to the proposed end. It gives both the greatest values of the equations and their arguments: in furnishing the latter it chiefly tends to improve the Lunar Tables; for the coefficients are most accurately determined by observation.

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* We cannot have a surer guide than the *computed* exactness: but it will be oftentimes easy to see that terms, when they are multiplied by certain powers and products of the eccentricities and inclinations, must become too minute for any practical purposes of exactness. In such cases a formal computation, (oftentimes a troublesome one) may be dispensed with.

† See Mason's Lunar Tables: the Tables in the third Volume of Vince's *Astronomy*: and *Tables de la Lune, par Burg.*

The arguments of equations that have, like the Variation and Evection, *short* periods, may be determined by the scientific examination of observations alone. There is no great difficulty in this: the difficulty is to detect, without the aid of theory, inequalities of *long* periods. Take, for instance, that equation which was discovered by Laplace, and by which, within these few years, the Lunar Tables have been improved. It is at least problematical whether, by mere observation alone, this equation, whose period is one hundred and eighty-five years, would ever have been detected. The same may be said concerning the *Secular Equation* (see p. 182.) (in fact, a periodical equation of an extremely long period) discovered by the same Author. Previously to their being discovered, Astronomers were much embarrassed with certain Anomalies in the mean motion: for the secular inequality, and any inequality that slowly passes, by minute degrees, from its first increase or decrease, to its state of maximum or minimum, must necessarily blend itself with the mean motion, and perplex its determination.

The terms that represent the value of u (see pp. 180, &c.) expound the equations by which it is necessary to correct the Moon's mean parallax in order to obtain her true; and, in the present Chapter, we have given the mathematical explanation of the equations that serve to correct the Moon's mean longitude. It remains to find the Moon's latitude, and, since the latitude as well as the radius vector and longitude is affected by the disturbing force, to find the terms that expound the inequalities in latitude: that is to be done by solving the differential equation [c] of p. 95. Now, setting aside the labour of computation, this is a matter of little difficulty, since the equation is similar to the equation which has been already integrated.

In the present Chapter we have spoken of the terms, that form the expression for the longitude, and of the corresponding *equations*, which they expound, as belonging to the Lunar Theory. That is, indeed, in *Physical Astronomy*, the theory of the greatest importance. But, it is plain, that inequalities and their corresponding equations, similar to the Lunar, will exist for every case of planetary disturbance: for Venus disturbed by the action of the Earth, and one of Jupiter's satellites disturbed by the action of another; and, with still stricter analogy, for any of the Satellites of Jupiter and Saturn disturbed by the Sun. There will be found to belong to these cases, equations the same, in the form of their arguments, as the *Variation*, *Evection*, and *Annual Equation:* not, indeed, so denominated, since the above terms have been appropriated, chiefly for historical reasons, to the Lunar Theory; nor always deserving to be distinguished on account of their magnitude: since the equations corresponding * to the largest in the Lunar Theory are not necessarily the largest in an instance of the Planetary Theory: the magnitudes of the terms expounding equations depend, often as we have seen (seen pp. 236, &c.) on the proportion between the mean motions of the disturbed and disturbing body; which must vary with the instance.

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* By this term is meant the equations that have similar arguments; $2 \cdot (♀ - ♃)$, for instance, in the case of Venus disturbed by Jupiter, is the argument corresponding to that of the Lunar Variation.

CHAP. XV.

On the Integration of the Equation on which the Moon's Latitude depends. Formation of Equations correcting the Latitude. Regression of the Nodes. Secular Equation of the Regression.

If s be the tangent of latitude,

γ the tangent of the inclination of the plane of the orbit,

θ the longitude of the node,

$g-1$ the regression of the node*,

then, when no disturbing force acts, the finite equation

$$s=\gamma\,.\,\sin.(g\,v-\theta),$$

is the integral of the equation [γ] of p. 96.

When a disturbing force acts, of which the resolved parts are, P, S and T, the quantity s must be determined by the integration of the equation [c] of p. 95. which equation, reduced as the equation [b] was in page 138., is

$$\left(\frac{d^2 s}{d v^2}+s\right)\left(1+\frac{2}{h^2}\int\frac{T d v}{u^3}\right)+\frac{1}{h^2 u^3}\,.\,T\frac{d s}{d v}+\frac{S}{h^2 u^3}-\frac{P s}{h^2 u^3}=0.$$

This equation is to be integrated exactly as the equation [b] has been in the preceding pages. The several parts, such as

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* To suppose, in this place, a regression, and to substitute an arbitrary quantity to represent it, is to anticipate a result; but the violation of the order of legitimate deduction is very slight, since it can be very easily shewn, (for what can be more simple than the reasonings of the tenth and eleventh Corollaries of the eleventh Section of the *Principia*), that the nodes cannot remain at rest: whether the motion be progressive or regressive does not affect the assumption of the term.

$T \cdot \frac{ds}{dv}$, &c. are to be deduced by first assuming for s its imperfect value (such as belongs to it in an undisturbed system), and then by correcting the results on the supposition that s varies, or has a variation such as δs. The method will be better understood by being exemplified.

First,

$$\textit{Value of } \frac{1}{h^2 u^3} \cdot T \cdot \frac{ds}{dv}.$$

Assume for s that value which it would have, were there no disturbing force: then,

$$s = \gamma \sin.(gv - \theta),$$

$$\text{and } \frac{ds}{dv} = g\gamma \cos.(gv - \theta).$$

Now, (see p. 60.) $\frac{T}{u^3} = -\frac{3m'u'^3}{2u^4} \sin. 2\omega$

$$= -\frac{3Ka}{2} \cdot \frac{(1 + e' \cdot \cos. c'mv)^3}{(1 + e \cdot \cos. cv)^4} \cdot \sin. 2\omega$$

$$= -\frac{3Ka}{2}(1 + 3e' \cdot \cos. c'mv - 4e \cdot \cos. cv) \cdot \sin. 2\omega,$$

excluding the terms that involve the products ee', &c. and the second and higher powers of e and e'. Substitute now for $\sin. 2\omega$ that part of its value which is contained in the first, second, fourth and fifth lines of p. 166. and there will result, combining, according to Trigonometrical formulæ, $\cos. gv - \theta$, and $1 + 3e' \cos. c'mv - 4e \cos. cv$,

$$\frac{1}{h^2} \cdot \frac{T}{u^3} \cdot \frac{ds}{dv} =$$

$$-\frac{3Ka}{4h^2} \cdot g\gamma \left\{ \begin{array}{c} 2\cos.(gv - \theta) \\ -4e \cdot \cos.(gv - cv - \theta) \\ -4e \cdot \cos.(gv + cv - \theta) \\ +3e' \cdot \cos.(gv - c'mv - \theta) \\ +3e' \cdot \cos.(gv + c'mv - \theta) \end{array} \right\} \times$$

$$\times \left\{ \begin{array}{l} \quad \sin.\,(2\,v - 2\,m\,v) \\ -\,2\,m\,e\,.\,\sin.\,(2\,v - 2\,m\,v - c\,v) \\ +\,2\,m\,e\,.\,\sin.\,(2\,v - 2\,m\,v + c\,v) \\ +\,2\,e'\,.\,\sin.\,(2\,v - 2\,m\,v - c'\,m\,v) \\ -\,2\,e'\,\sin.\,(2\,v - 2\,m\,v + c'\,m\,v) \end{array} \right\}$$

The combination of these factors will produce twenty-five rectangles, each of the form sin. A . cos. B, consequently, by the development of the rectangles, (since $\sin.\,A\,.\,\cos.\,B = \frac{1}{2}\,[\sin.\,(A+B) + \sin.\,(A-B)]$ fifty terms, each the sine of an arc, supposing each arc different: but this is not the case: several of the terms, being the sines of the same arcs, with equal coefficients of opposite signs, destroy each other: for instance, the second and third terms of the first series of factors of the preceding expressions combining with the second and third terms of the last series of factors, ought, were there no relation between the arcs, to produce eight sines of arcs: but they produce only four: since the second term of the last series, combining with the second and third of the first produces, with other, two terms equal

$$+\,4\,m\,e^2 \left\{ \begin{array}{l} \sin.\,(2v - 2\,m\,v + g\,v + \theta) \\ +\sin.\,2v - 2\,m\,v + g\,v - \theta) \end{array} \right\},$$

and the third term of the last series combining with the two same terms of the first, produces the same terms but with coefficients equal $-\,4\,m\,e^2$. The same kind of reduction will take place on the combination of the fourth and fifth terms of the last series of factors, with the fourth and fifth of the first series. A reduction not unlike the preceding, but less in degree, will take place amongst those terms that involve the sines of the same arcs with unequal coefficients.

These considerations are useful in abridging the process of computation when it is intended to be carried to great exactness: which is not the case at present, since the terms that involve c^2, e'^2, $e\,e'$, are purposely excluded. If we retain then no terms that contain powers of e, e', beyond the first, and make, for that same reason $h^2 = a_{,}$, we shall have

$$\frac{1}{h^2 u^3} \cdot T . \frac{ds}{dv} =$$

$$-\frac{3K}{4} \cdot \frac{a}{a_{,}} . g\gamma \left\{ \begin{array}{l} \sin. (2v - 2mv - gv + \theta) \\ + \sin. (2v - 2mv + gv - \theta) \\ - 2 . (1 + m) e . \sin. (2v - 2mv + gv - cv - \theta) \\ + 2 . (1 + m) e . \sin. (gv + cv - 2v + 2mv - \theta) \\ - 2 . (1 - m) e . \sin. (2v - 2mv - gv + cv + \theta) \\ - 2 . (1 - m) e . \sin. (gv + cv + 2v - 2mv - \theta) \\ + \frac{7}{2} e' . \sin. (2v - 2mv - gv - c'mv + \theta) \\ + \frac{7}{2} e' . \sin. (2v - 2mv + gv - c'mv - \theta) \\ - \frac{e'}{2} . \sin. (2v - 2mv - gv + c'mv + \theta) \\ - \frac{e'}{2} . \sin. (2v - 2mv + gv + c'mv - \theta) \end{array} \right\}$$

We will next find

$$\frac{S}{h^2 u^3} - \frac{Ps}{h^2 u^3},$$

which (see pp. 65, 169.), equals

$$\frac{s}{h^2 u (1 + s^2)^{\frac{3}{2}}} + \frac{m' s u'^3}{h^2 u^4} + \frac{3 m' s u'^4}{h^2 u^5} \cos. (v - v')$$

$$- \frac{s}{h^2 u (1 + s)^{\frac{3}{2}}} + \frac{m' u'^3 s}{2 h^2 u^4} + \frac{3 m' u'^3 s}{2 h^2 u^4} \cos. (2v - 2v').$$

In the Lunar Theory $\frac{u'}{u} = \frac{r}{r'}$ is very small, if therefore we reject the last term in the first line of the preceding value, since it contains $\frac{s u'^4}{u^5}$, and reduce the other terms, we shall have

$$\frac{S}{h^2 u^3} - \frac{Ps}{h^2 u^3} = \frac{3 m' u'^3 s}{2 h^2 u^4} + \frac{3 m' u'^3 s}{2 h^2 u^4} \cos. (2v - 2v').$$

First, with regard to the first term: if we exclude the terms that involve the squares, cubes, &c. of e, e', &c. we shall have

$$\frac{3\,m'\,u'^3\,s}{2\,h^2\,u^4} =$$

$$\left(= \frac{3\,K\,a}{2\,a_{,}} \cdot \gamma \cdot \sin.(g\,v - \theta)\,(1 - 4\,e \cos.\,c\,v)\,(1 + 3\,e' \cos.\,c'\,m\,v) \right)$$

$$\frac{3\,K}{2} \cdot \frac{a}{a_{,}} \cdot \gamma \left\{ \begin{array}{l} \sin.(g\,v - \theta) \\ -\,2\,e \,.\, \sin.(g\,v + c\,v - \theta) \\ -\,2\,e \,.\, \sin.(g\,v - c\,v - \theta) \\ +\,\dfrac{3\,e'}{2} \sin.(g\,v + c'\,m\,v - \theta) \\ +\,\dfrac{3\,e'}{2} \sin.(g\,v - c'\,m\,v - \theta) \end{array} \right\}$$

If the terms involving e^2, e'^2, &c. had been retained, then

$$a^4\,(1 + e^2 - \gamma^2 - 4\,e \,.\, \cos.\,c\,v),$$

instead of $a^4\,(1 - 4\,e \,.\, \cos.\,c\,v)$, would have represented the value of u^{-4}, and the value of u'^3 would have been represented by

$$\frac{1}{a'^3}\left(1 + \frac{3\,e'^2}{2} + 3\,e' \cos.\,c'\,m\,v\right)$$

and the value of $\frac{1}{h^4}$ by

$$\frac{1}{a_{,}}\,(1 + e^2 + \gamma^2);$$

in which case, the coefficient of the first term within the brackets, namely, $\sin.(g\,v - \theta)$, would have become

$$\frac{3\,K}{2} \cdot \frac{a}{a_{,}} \cdot \gamma \left(1 + 2\,e^2 + \frac{3\,e'^2}{2}\right),$$

and in the Lunar Theory, where great exactness is required (and is indeed practised), it is necessary to attend to such terms, especially on account of e'^2, which varies from the disturbing force of the planets, and on which the *Secular Equation of the Node*, as well as the Secular Equations of the Mean Motion, and Progression of the Apogee, depend.

In order to obtain the second term of the preceding expression of p. 246. (l 3. from bottom), it is merely necessary to multiply the last series by the value of $\cos.(2v - 2\,v')$, and, in the present case, by that part of its value in p. 166, which is contained in the first, second, fifth and sixth lines: if this be done, there will result

$$\frac{3\,m'\,u'^3\,s}{2\,h^2\,u^4}\cos.\,2\,\omega =$$

$$\frac{3\,K}{4}\cdot\frac{a}{a_{,}}\cdot\gamma\left\{\begin{array}{c}
\sin.\,(2\,v - 2\,m\,v + g\,v - \theta) \\
-\sin.\,(2\,v - 2\,m\,v - g\,v + \theta) \\
-\,2\,.\,(1+m)\,e\,.\,\sin.\,(2\,v - 2\,m\,v + g\,v - c\,v - \theta) \\
-\,2\,.\,(1+m)\,e\,.\,\sin.\,(g\,v + c\,v - 2\,v + 2\,m\,v - \theta) \\
+\,2\,.\,(1-m)\,e\,.\,\sin.\,(2\,v - 2\,m\,v - g\,v + c\,v + \theta) \\
-\,2\,.\,(1-m)\,e\,.\,\sin.\,(g\,v + c\,v + 2\,v - 2\,m\,v - \theta) \\
-\,\frac{7}{2}\,e'\,.\,\sin.\,(2\,v - 2\,m\,v - g\,v - c'\,m\,v + \theta) \\
+\,\frac{7}{2}\,e'\,.\,\sin.\,(2\,v - 2\,m\,v + g\,v - c'\,m\,v - \theta) \\
+\,\frac{e'}{2}\,.\,\sin.\,(2\,v - 2\,m\,v - g\,v + c'\,m\,v + \theta) \\
-\,\frac{e'}{2}\,.\,\sin.\,(2\,v - 2\,m\,v + g\,v + c'\,m\,v - \theta)
\end{array}\right\}$$

What remains to be done is to find the value of

$$\left(\frac{d^2\,s}{d\,v^2} + s\right)\frac{2}{h^2}\int\frac{T\,d\,v}{u^3}.$$

Now, since

$$s = \gamma\,\sin.\,(g\,v - \theta),$$

$$\frac{d\,s}{d\,v} = g\,\gamma\,\cos.\,(g\,v - \theta),$$

$$\text{and }\frac{d^2 s}{d\,v^2} = -\,g^2\,\gamma\,.\,\sin.\,(g\,v - \theta) = -\,g^2\,s;$$

$$\therefore\ \frac{d^2\,s}{d\,v^2} + s = (1 - g^2)\,s = (1 - g^2)\,\gamma\,.\,\sin.\,(g\,v - \theta).$$

If this last value be multiplied into the value of $\frac{2}{h^2}\int\frac{T\,d\,v}{u^3}$ given in pp. 154, 168, there will result

$$\left(\frac{d^2 s}{d v^2} + s\right) \frac{2}{h^2} \int \frac{T d v}{u^3} =$$

$$\frac{3K}{4} \cdot \frac{a}{a_{,}} \cdot \frac{g^2 - 1}{1 - m} \left\{ \begin{array}{l} \sin.\ (2v - 2mv - gv + \theta) \\ -\sin.\ (2v - 2mv + gv - \theta) \\ + \&c. \end{array} \right\}$$

of which terms it will be sufficient, on the score of exactness, to retain the first.

The differential equation, if we now collect its several terms, will be of this form

$$\begin{aligned}
0 = {} & \frac{d^2 s}{dv^2} + s \\
& + \frac{3K}{2} \cdot \frac{a}{a_{,}} \left(1 + 2e^2 + \frac{3e'^2}{2}\right) \gamma . \sin.\ (gv - \theta) \\
& - \frac{3K}{4} \cdot \frac{a}{a_{,}} \left(1 + g + \frac{1 - g}{1 - m}\right) \gamma \sin.\ (2v - 2mv - gv + \theta) \\
& + \frac{3K}{4} \cdot \frac{a}{a_{,}} (1 - g) \gamma \sin.\ (2v - 2mv + gv - \theta) \\
& - 3K \cdot \frac{a}{a_{,}} e \gamma \sin.\ (gv + cv - \theta) \\
& - 3K \frac{a}{a_{,}} e \gamma \sin.\ (gv - cv - \theta) \\
& + \frac{3K}{2} \cdot \frac{a}{a_{,}} (g + 1)(1 - m) e \gamma \sin.\ (2v - 2mv - gv + cv + \theta) \\
& + \frac{3K}{2} \cdot \frac{a}{a_{,}} (g - 1)(1 + m) e \gamma \sin.\ (2v - 2mv + gv - cv - \theta) \\
& + \frac{3K}{2} \cdot \frac{a}{a_{,}} (g + 1)(1 + m) e \gamma \sin.\ (2v - 2mv - gv - cv - \theta) \\
& + \frac{3K}{2} \cdot \frac{a}{a_{,}} (g - 1)(1 - m) e \gamma \sin.\ (2v - 2mv + gv + cv - \theta) \\
& + \frac{9K}{4} \cdot \frac{a}{a_{,}} e' \gamma \sin.\ (gv + c'mv - \theta) \\
& + \frac{9K}{4} \cdot \frac{a}{a_{,}} e' \gamma \sin.\ (gv - c'mv - \theta) \\
& + \frac{3K}{4} \cdot \frac{a}{a_{,}} \frac{(g + 1)}{2} e' \gamma \sin.\ (2v - 2mv - gv + c'mv + \theta) \\
& - \frac{3K}{4} \cdot \frac{a}{a_{,}} \cdot \frac{7}{2} (g + 1) e' \gamma . \sin.\ (2v - 2mv - gv - c'mv + \theta) \\
& + \&c.
\end{aligned}$$

This equation is similar to the equation of p. 156, and admits of a similar integration; of such, indeed, as was explained in pp. 100, &c. The integration gives the value of s, and, in the case before us, will express it by a series of terms of which the arguments are the same as in the preceding differential equation, namely, $gv - \theta$, $2v - 2mv - gv - \theta$, $gv + cv - \theta$, &c. &c., and the coefficients, the corresponding coefficients in the differential equation divided respectively by

$$g^2 - 1, \quad (2 - 2m - g)^2 - 1, \quad (g + c)^2 - 1, \text{ \&c.}$$

(see p. 100.)

The terms representing the value of s mathematically expound, as in the preceding cases of the values of u and v, certain *equations* that serve to correct the latitude. Under a merely mathematical view, the terms, and, consequently, the equations, are infinite in number. But it is sufficient to retain a few: those that are, on account of their magnitude, eminent above the rest. And, as in the former cases, when the parallax and longitude were determined, so in this it may be shewn why some terms are much larger than others. The two first terms, for instance, that by integrating the preceding differential equation, will express s, are

$$\frac{3Ka}{a_{,} \cdot (g^2 - 1)} \left(1 + 2e^2 + \frac{3e'^2}{2}\right) \gamma \sin. (gv - \theta)$$

$$-\frac{3K}{4} \cdot \frac{a}{a_{,}} \cdot \frac{1}{(2-2m-g)^2 - 1} \left(1 + g + \frac{1-g^2}{1-m}\right) \gamma \sin. (2v - 2mv - gv + \theta).$$

Now each of these terms is large, by reason of the smallness of its denominator. The first expounds the *Equation*, the argument of which is termed the *Argument of Latitude:* the second expounds the principal equation of latitude, and which, from mere observation alone, was discovered by *Tycho Brahé*.

The coefficient of the first term may be used for determining the *Regression of the Lunar Nodes*, just as the coefficient of cos. cv in the value of u, was used by Clairaut for determining the *Progression of the Apogee:* thus the value of s in the undisturbed system is

$$s = \gamma \sin. (gv - \theta).$$

Equate this with

$$\frac{3\,K\,a}{2\,a_{,}\,(g^2-1)}\left(1+2\,e^2+\frac{3\,e'^2}{2}\right)\gamma\ \sin.\,(g\,v-\theta),$$

and there results

$$g=\sqrt{\left[1+\frac{3\,K\,a}{2\,a_{,}}\left(1+2\,e^2+\frac{3\,e'^2}{2}\right)\right]}.$$

For the purpose of deducing the arithmetical value of g, we have

$$e'=.016803,$$
$$e=.054873,$$
$$\frac{K\,a}{a_{,}}=.005595\,;$$

whence $g = 1.0042$, nearly,
and $g - 1 = .0042$.

This, for reasons such as are assigned in Chap. XIII. must be an inexact value: but, it is not so enormously inexact, as the first resulting value $(1-c)$ of the progression of the apogee: since, by a repeated process, $g-1=.0040105$. The cause of the inexactness of the first resulting value, and the means of correcting it have been fully explained in the preceding pages: since what was there observed on the method of deducing the progression of the apogee is strictly applicable to the present case.

It is difficult to find any very simple mode of treating the Progression of the Apogee. Clairaut's method is as obvious as any other; and, as it has been observed, the preceding method in the text is analogous to it. But that, when the Regression of the Nodes is the object of investigation, is far from being the most simple method. Newton's is much more simple; and, which is a rare excellence, it at once shews the regression to be an obvious effect of the disturbing force, and affords the means of computing its quantity. This kind of excellence, however, depends in a great degree, on the nature of the subject of research, and consequently, in *Physical Astronomy*, is very limited. It cannot be expected to be found, (as the very terms, indeed, signify) in abstruse subjects. But the method described in p. 250. although less simple than Newton's, has yet its peculiar advantages. $g - 1$, which expresses the *mean Regression*, is equal to

$$\sqrt{\left[1 + \frac{3Ka}{2a_{,}}\left(1 + 2e^2 + \frac{3e'^2}{2}\right)\right]} - 1.$$

Now, as it has been before stated, the eccentricity (e') of the Solar Orbit is rendered variable by the action of the planets. It is subject to a *Secular Equation;* consequently the mean Regression of the Lunar Nodes is also subject to a Secular Equation. A similar inference was made in p. 181. from the value of $1 - c$, relative to the Secular Equation of the Progression of the Lunar Apogee; and such inferences are more easily made from Clairaut's than from Newton's method.

Laplace has still a different method, but one resembling that which he uses for determining the *Progression* (pp. 212. 223. 2nde Partie, Liv. VII. *Mec. Cel.*): and it follows also immediately from this method that the Regression of the Nodes is subject to a Secular Equation.

The preceding pages relate principally to the periodical inequalities of the Moon: those inequalities which prevent her parallax, longitude, and latitude, being what they would be were the sole force acting on her the Earth's attraction. But the Progression of the Apogee, and the Regression of the Nodes, belong to a distinct class of inequalities, such as affect the very orbit itself, its dimensions and position in space. These inequalities are technically denominated the *Variations of the Elements.* One element is the position or longitude of the Apogee: another the longitude of the Node: the *Variation* of the former is the *Progression of the Apogee,* of the latter the *Regression of the Node:* and these have been treated of: at least, their mean quantities and the *Secular Inequalities* affecting them. A third element is the eccentricity of the Lunar Orbit. This, Newton, in the eleventh Section of his *Principia,* shewed to be subject to change from the Sun's disturbing force: and, on grounds and by considerations similar to those which we used in speaking of the cause of the *Evection.* But Newton gave no method of computing its quantity as he did in the case of the nodes. Indeed, the variation of the eccentricity is not, like the progression of the apogee, and the regression of the nodes, a distinct phenomenon: it is combined with, and influences, other inequalities. It could not, therefore, by the agreement of its computed and observed quantity, readily serve, like the two other variations, to confirm the Law of Gravitation.

A fourth element is the inclination of the plane of the Lunar Orbit. In his eleventh Section, Newton shewed the *Variation* of this element to be a necessary consequence of the Sun's disturbing force: and, in the third Book of the *Principia* he computes its law and quantity. A fifth element is the semi-axis of the Moon's Orbit, or her mean distance. If we look to phenomena that element has no variation. Newton, therefore, in the third Book of the *Principia*, could derive no confirmation of his principle and Law of Attraction from the agreement of its computed and observed quantity. The Moon's mean motion was found to be invariable, and therefore her mean distance would be so. Still it is remarkable that, the other elements varying, this should remain constant: that it should be so, both when the disturbing force acted, and when it did not. This is, in itself, a kind of phenomenon which requires an explanation. It is necessary to shew, at least for the purposes of curious inquiry, that a disturbing force can make no alteration in the mean distance. There cannot be a less self-evident proof of the Sun's attraction than the invariability of that element. It affords on first views, if any thing, a presumption against the principle of *universal* attraction. The invariability therefore, of the mean distance is a thing to be established on Newton's principles: and being established, is, at least, equally a proof of their truth as the *Variability* of the Apogee.

Newton has given nothing on this subject in his *Principia:* it was not to be expected that the founder of a great system should have had leisure to attend to all its details. Investigations of a nature so abstruse as those that have been just described, would, during the establishing of a new system, be postponed, and made to give place to others more obvious and important. The mathematicians, however, who succeeded Newton had leisure to attend to this subject.

They have (and this is one of the usual effects of progressive Science) considered the variations of the elements under a general point of view, and reduced the expressions of their values to six similar differential formulæ. From one of these it results that the major-axis of a planet's orbit is subject to no secular inequality: and consequently that the planets, notwithstanding their mutual action, will constantly preserve the same distances from the Sun.

This is one of the points of the *permanence* or *stability of the Planetary System;* a subject of considerable importance and interest: and, the periodical inequalities of parallax, longitude, and latitude, having been investigated, this might now seem to be the proper place to consider the changes produced by disturbing forces in the dimensions and position of a planet's orbit. And so indeed it would be, were the preceding solution of the *Problem of the Three Bodies* immediately applicable to the case of any planet revolving round the Sun and disturbed by another planet. But the fact is otherwise. The instance, indeed, of Venus revolving round the Sun and disturbed by the Earth resembles, in its general character, that of the Moon revolving round the Earth and disturbed by the Sun. To each case belongs the same differential equation, and the same method of integration. There is, however, a difference which is to be found in the detail, and which is entirely mathematical. We will explain in what it consists.

The value of P which is given in page 60, and which was used in the succeeding series of solutions was derived from the general value of page 57. by expanding $\frac{1}{y^3}$ (see p. 59.) and by rejecting, in its development, all terms after the second. Now this rejection is founded on the minuteness of $\frac{r}{r'}$, and $\frac{r}{r'}$ in the Lunar Theory is $\frac{1}{400}$: but in the case of Venus disturbed by the Earth $\frac{r}{r'} = .723332$: therefore, $\left(\frac{r}{r'}\right)^2$, $\left(\frac{r}{r'}\right)^3$, &c. cannot be rejected: and consequently the analytical expression for P cannot remain the same as it was in the Lunar Theory. The processes and results, therefore, of Chapters IX, X, &c. will at least require some modification, or, as we shall soon see, the invention of new methods.

In the less simple expression for P, then, the planetary theory seems more complicated than the Lunar; but in other respects it is much less so. The chief cause is, the smallness of the disturbing force of any planet compared with the Sun's disturbing force. The Lunar perturbations require thirty equations, but three are sufficient to express the inequalities of Venus produced by Jupiter's action. For the simple cases, then, that occur in the Planetary Theory, the apparatus of formulæ and processes, that has been used in determining the Moon's place, is too cumbrous and complicated. The formulæ will serve indeed, as Clairaut in the *Memoirs of the Academy of Sciences* for 1754. pp. 521, &c. made them to serve, for determining the Earth's place disturbed by the Moon, Jupiter and Venus. But the method is not an expeditious one: and the first result, the expression of the mean anomaly in terms of the true, is not the main object of investigation. That object is, the true longitude in terms of the mean: and in order to obtain it, the *Reversion of Series* (see pp. 228, &c.), an operation of some difficulty, must be used.

A shorter method of solution, for the more simple cases, has been obtained by abandoning the equations [a], [b] of p. 95. for equations expressing r and its differentials, v and its differentials, in terms of $n\,t$, $n\,d\,t$, and other quantities. The equations of solution, when obtained, are, indeed, more easy of application than the former; but they are deduced by less obvious processes.

We have stated then two points of distinction between the Lunar and Planetary Theories, and which entitle the latter to a separate discussion.

The case that bears the strictest analogy to the theory of the Lunar perturbations is that of a Satellite of Jupiter disturbed by the Solar attraction. The problems in every particular are precisely the same. The Satellite's mean longitude, in order that its true may be found, must be corrected by the three equations of the *Variation*, the *Evection*, and the *Annual Equation*. But these are very small corrections, and the other equations that correspond to the Lunar exist only theoretically, and are insignificant when numerically expounded.

The instances that resemble, but less closely, the Moon disturbed by the Sun are, as it has been already stated, Venus disturbed by the Earth, or by Mars, or by Jupiter; or, Mars disturbed by Jupiter; or, one of Jupiter's Satellites disturbed by another more distant Satellite. To these cases, that solution of the Problem of the Three Bodies which was used in the Lunar Theory does not immediately (see p. 254.) apply. We may presume also that it will not, without some modification, apply to the perturbation of a planet disturbed by another, the orbit of which is *interior* to that of the former. For still less closely than either of the two preceding instances, does that of the Earth disturbed by the Moon, or by Venus, resemble the Moon disturbed by the Sun. It would be a loss of time to attempt to describe, in general terms, in what the difference consists. We must descend into the details and view it nearly.

The succeeding Chapters then, will be specially appropriated to the Planetary Theory, which, in many respects, is less complicated than the Lunar; and, in some of its instances, we shall find ourselves thrown back on the most simple cases of the Problem of the Three Bodies. The Planetary Theory, however, is not without its peculiar difficulties.

CHAP. XVI.

ON THE PLANETARY THEORY.

Differential Equation for determining the Radius Vector: Expression for R: *its development into a Series of Cosines of Multiple Arcs. Conditions on which the Convergency of such Series depends. Application of the Differential Equation to the Investigation of the Perturbations in the Radius Vector and Longitude of the Earth by the Moon's Action.*

THE first object in this Chapter is to obtain differential equations from which the radius vector and longitude may be obtained more concisely than from the equations [*a*], [*b*], of p. 95. that have been employed in the Lunar Theory.

If in the equations [4], [5], of p. 92. we suppose the body's latitude to be nothing, and substitute instead of *P* and *T*, their values such as are given in p. 66, namely,

$$\frac{\mu}{r^2} + \frac{dR}{dr}, \text{ and } \frac{dR}{dv},$$

there will result

$$d^2 r - r\,d^2 v + \left(\frac{\mu}{r^2} + \frac{dR}{dr}\right) d\,t^2 = 0 \ldots\ldots\ldots [1],$$

$$2\,dr\,.\,dv + r\,d^2 v + \frac{dR}{r.dv}\,.\,d^2 t = 0 \ldots\ldots\ldots [2].$$

Multiply [1] by dr, and [2] by $r\,dv$, and add the results, then

$$0 = \left\{\begin{array}{l} dr\,.\,d^2 r + r\,dr\,.\,dv^2 + r^2 dv\,.\,d^2 v + \mu\frac{dr}{r^2}\,dt^2 \\ \qquad + \left(\frac{dR}{dr}\,dr + \frac{dR}{dv}\,dv\right) dt^2 \end{array}\right\}$$

from which equation integrated and corrected there results

$$0 = \frac{1}{d t^2}(d r^2 + r^2 d v^2) - \frac{2\mu}{r} + \frac{\mu}{a} + 2\int d R \ldots\ldots\ldots\ldots [3]$$

Since, $\int\left(\frac{dR}{dr}\,dr + \frac{dR}{dv}\,dv\right) = \int dR.$*

For the purpose of eliminating dv from this equation [3], substitute, instead of $r^2\,dv^2$, that value which the equation [1] multiplied by r will give, then

$$\frac{1}{dt^2}\left[(dr)^2 + r d^2 r\right] - \frac{\mu}{r} + \frac{\mu}{a} + 2\int dR + r\frac{dR}{dr} = 0.$$

Now, $(dr)^2 + r\,d^2 r = \frac{1}{2}\,d^2(r^2)$,

and if δr be made to signify that variation of r which arises from the disturbing force, then, on neglecting $(\delta r)^2$ and the products of m' (see p. 66.) and δr, there will result

$$\frac{1}{2\,dt^2}\left[d^2(r^2) + d^2(2r\,\delta r)\right] - \frac{\mu}{r} + \frac{\mu r\,\delta r}{r^3} + \frac{\mu}{a} + 2\int dR + r\frac{dR}{dr} = 0.$$

But, when there is no disturbing force, and r, accordingly, has its elliptical value,

$$\frac{1}{2}\,d^2(r^2) - \frac{\mu}{r} + \frac{\mu}{a} = 0.$$

Hence, since $\frac{1}{r^3} = \frac{1}{(a+\delta a)^3} = \frac{1}{a^3} - \frac{3\,\delta a}{a^4}$, (representing by δa that variation of a which is due to the disturbing force), we have

$$d^2 . \frac{r\,\delta r}{dt^2} + N^2 . r\,\delta r + 2\int dR + r . \frac{dR}{dr} = 0,$$

in which equation,

$$N^2 = \frac{\mu}{a^3}\left(1 - \frac{3\,\delta a}{a}\right) = n^2\left(1 - \frac{3\,\delta a}{a}\right).$$

We must now consider whether it is possible to express

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* *Anal. Calc.* p. 78.

$2\int dR + \frac{r\,dR}{dr}$ by a series of cosines such as A cos. $q\,n\,t$: for then we should be able immediately to integrate the equation by the method of pp. 97, &c.

The value of R (see p. 66.) on making $s = 0$, and $\rho = r$ is thus expressed,

$$R = \frac{m'\,r}{r'^4}\cos.(v' - v) - \frac{m'}{\sqrt{[r'^2 - 2\,r\,r'.\cos.(v' - v) + r^2]}}.$$

Now, supposing ϵ, ϵ' to denote the epochs at which the bodies are in the perihelia of their orbits, we have (see p. 32.)

$$\begin{aligned}
v &= n\,t + \epsilon + 2e.\sin.(n\,t + \epsilon - \pi) + \text{\&c.}\\
v' &= n'\,t + \epsilon' + 2e'.\sin.(n'\,t + \epsilon' - \pi'),\\
r &= a - a\,e.\cos.(n\,t + \epsilon - \pi) + \text{\&c.}\\
r' &= a' - a'e'.\cos.(n'\,t + \epsilon' - \pi') + \text{\&c.}
\end{aligned}$$

let us begin with a simple case, and suppose the orbits to be so nearly circular that the terms involving e, e', may be neglected, then

$$R = \frac{m'\,a}{a'^2}.\cos.(n'\,t - n\,t + \epsilon' - \epsilon)$$

$$- \frac{m'}{\sqrt{(a'^2 - 2\,a\,a'.\cos.(n'\,t - n\,t + \epsilon' - \epsilon) + a^2)}},$$

which may always, whatever are the relative magnitudes of a, a', be expanded into a series of terms such as

$$A^{(0)} + A^{(1)}.\cos.(n'\,t - n\,t + \epsilon' - \epsilon) + A^{(2)}.\cos.2\,(n'\,t - n\,t + \epsilon' - \epsilon)$$
$$+ \text{\&c.}$$

For, make $\varpi = n'\,t - n\,t + \epsilon' - \epsilon$,

$$\alpha = \frac{a}{a'},$$

and $y = \sqrt{(a'^2 - 2\,a\,a'\cos.\varpi + a^2)}$;

then

$$\frac{1}{y^{4s}} = \frac{1}{a'^{4s}}(1 - 2\,\alpha\cos.\varpi + \alpha^2)^{-2s}$$

$$= \frac{1}{a'^{4s}}\left[1 - \alpha\left(x + \frac{1}{x}\right) + \alpha^2\right]^{-2s},$$

$$\left(\text{making } x + \frac{1}{x} = 2 \cos.\, \omega\right)$$

$$= \frac{1}{a'^{4s}} (1 - \alpha x)^{-2s} . \left(1 - \frac{\alpha}{x}\right)^{-2s}.$$

Now $(1 - \alpha x)^{-2s} = 1 + 2s . \alpha x + \frac{2s . (2s+1)}{1 . 2} \alpha^2 x^2$

$$+ \frac{2s . (2s+1)(2s+2)}{1 . 2 . 3} \alpha^3 x^3 + \&c.$$

$$\left(1 - \frac{\alpha}{x}\right)^{-2s} = 1 + \frac{2s\alpha}{x} + \frac{2s . (2s+1)}{1 . 2} \frac{\alpha^2}{x^2}$$

$$+ \frac{2s . (2s+1)(2s+2)}{1 . 2 . 3} \frac{\alpha^3}{x^3} + \&c.$$

and their product is equal to

$$1 + (2s)^2 . \alpha^2 + \left(\frac{2s . (2s+1)}{1 . 2}\right)^2 . \alpha^4$$

$$+ \left(\frac{2s . (2s+1)(2s+2)}{1 . 2 . 3}\right)^2 \alpha^6 + \&c.$$

$$+ \left\{ \begin{array}{l} 2s\alpha + \frac{(2s)^2 . (2s+1)}{1^2 . 2} \alpha^3 \\ + \frac{(2s)^2 . (2s+1)^2 . (2s+2)}{1^2 . 2^2 . 3} \alpha^5 + \&c. \end{array} \right\} \left(x + \frac{1}{x}\right)$$

$$+ \left\{ \begin{array}{l} \frac{2s^2 . (2s+1)}{1 . 2} \alpha^2 + \frac{(2s)^2 . (2s+1) . (2s+2)}{1^2 . 2 . 3} \alpha^4 \\ + \frac{(2s)^2 . (2s+1)^2 . (2s+2)(2s+3)}{1^2 . 2^2 . 3 . 4} \alpha^6 + \&c. \end{array} \right\} \left(x^2 + \frac{1}{x^2}\right)$$

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[*] The coefficients of this product may, and rather more *regularly*, be thus expressed: (see *Mec. Anal.* 2de Partie, Sect. VII.).

$$1 + (2s)^2 . \alpha^2 + \left(\frac{2s . (2s+1)}{1 . 2}\right)^2 \alpha^4 + \&c.$$

$$+ \alpha\left(2s + \frac{2s}{1} . \frac{2s . (2s+1)}{1 . 2} \alpha^2 + \frac{2s . (2s+1)}{1 . 2} . \frac{2s(2s+1)(2s+2)}{1 . 2 . 3} \alpha + \&c.\right)$$

$$+ \alpha^2 \left(\frac{2s . (2s+1)}{1 . 2} + \frac{2s . (2s+1)}{1 . 2} . \frac{2s . (2s+1)(2s+2)}{1 . 2 . 3} \alpha^2 + \&c.\right)$$

$$+ \alpha^3 \left(\frac{2s . (2s+1)(2s+2)}{1 . 2 . 3} + \&c.\right).$$

but see *Trig.* p. 42.

$$x^2 + \frac{1}{x^2} = 2 \,.\, \cos.\, 2\,\omega,$$

$$x^3 + \frac{1}{x^3} = 2 \,.\, \cos.\, 3\,\omega,$$

$$\&c. \quad = \&c.$$

consequently,

$$\frac{1}{y^{4s}} = \frac{1}{a'^{4s}}\,(M + N \,.\, \cos.\, \omega + O \,.\, \cos.\, 2\,\omega + \&c.)$$

$M, \frac{N}{2}, \frac{O}{2}$, &c. representing the coefficients of 1, $x + \frac{1}{x}$, $x^2 + \frac{1}{x^2}$, &c. in the preceding product (p. 260.)

On account of the importance of the formula*, we have deduced its most general expression in which s may designate any number. In the instance that gave rise to the investigation,

$$4\,s = 1, \text{ and } 2\,s = \frac{1}{2},$$

and if this fraction be substituted, we shall have the same series as Laplace has given in p. 272. of his *Mec. Celeste.*

It appears then, if we regard merely the analytical expression and not the convergency of the series as dependent on the value of a, that R can always be expressed by a series such as

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[*] The development of $\frac{1}{y^{4s}}$, or of $(r'^2 - 2\,r\,r' \cos.\, \omega + r^2)^{-2s}$ was, during his researches on the perturbation of the planets, deduced by Lagrange and by the aid of impossible quantities. In this method he has been followed by Laplace and other authors (see *Mem Berlin,* 1781. p. 257. *Mec. Anal.* p. 142. *Acad. des Sciences,* 1785, p. 68. *Mec. Celeste,* tom. I. pp. 271, 272. Vince's *Astron.* vol. II. p. 191.) The demonstration in the text is obtained with an expedition quite equal to that which the use of imaginary symbols is able to confer.

$$A^0 + A^{(1)} \cos.(n't - nt + \epsilon' - \epsilon) + A^{(2)} . \cos. 2(n't - nt + \epsilon' - \epsilon) + \&c.$$

and consequently, $2\int dR + r\frac{dR}{dr}$, or, in this case, $2\int dR + a . \frac{dR}{da}$ can always be expressed by means of terms involving the cosines of arcs the multiples of $n't - nt + \epsilon' - \epsilon$; in which case (see pp. 259, &c.) the equation of p. 258. can be integrated by the method of pp. 97, &c.

We must now consider whether the method is an easy practical one: and that must depend, as it is plain, on the convergency of the series that expresses the value of R. If $\frac{r}{r'}$, or $\frac{a}{a'}$ be a small fraction, the series will quickly converge; and a few of its terms will, with sufficient exactness, represent its sum.

If the fraction $\frac{a}{a'}$, should be as small as it is $\left(=\frac{1}{400}\right)$ in the Lunar Theory, the series would converge so quickly that it would be sufficiently exact to retain its two first terms. There is no reason why we should not use the series when it has been once invented: but otherwise it would have been an useless refinement to have invented it for so simple a case. The binomial theorem see p. 59. immediately affords the proper result.

In the case last alluded to, a', which expresses the radius of the orbit of the disturbing planet, was the radius of the Sun's orbit: and, $\frac{a}{a'}$, being very small, the series converged very rapidly: but we may still obtain a converging series, if a', continuing to represent the radius of the orbit of the disturbing body, should be less than a: in other words, if the orbit of the disturbing body should be interior to that of the disturbed:

for, since $y = \sqrt{(a'^2 - 2aa' \cos. \omega + a^2)}$,

$$\frac{1}{y^{4s}} \text{ either } = \frac{1}{a'^{4s}}\left(1 - \frac{2a}{a'}\cos. \omega + \frac{a^2}{a'^2}\right)^{-2s},$$

$$\text{or } = \frac{1}{a^{4s}}\left(1 - \frac{2a'}{a}\cos. \omega + \frac{a'^2}{a^2}\right)^{-2s},$$

and consequently, the same form of development would belong to each case: and, if the first converged, a' being greater than a, the latter would converge, and ought to be used, a' being less than a: and, as it is plain, the convergency would be the same, if $\frac{a}{a'}$ should equal $\frac{a'}{a}$.

As far then as depends on the facility of computing R (and consequently of computing P and T, see pp. 66.) by the convergency of the series expressing it, the problem of Jupiter's perturbations by the action of Mars is equally easy with that of the perturbation of Mars by Jupiter.

Hence too, that application of the series, which in the case of the Earth, Moon and Sun, we stated (p. 262.) to be most simple, the Sun being the disturbing body, will be equally so, when the Moon becomes the disturbing body: and, since this is a case the most simple of any in the Theory of Perturbations, whether the orbit of the *third* body be without or within that of the disturbed, we will apply to it the formulæ of this Chapter. The results may then be compared with those which have already, on different principles, been previously (see Chap. VI.) obtained.

If we make $4s = 1$, $2s$ will $= \frac{1}{2}$,

and $\frac{1}{y}$ (see p. 260.) will $= \frac{1}{a}\left[1 + \left(\frac{1}{2}\right)^2 . \left(\frac{a'}{a}\right)^2 + \&c.\right]$

$$+ \frac{1}{a}\left[\frac{a'}{a} + 2 . \frac{1}{2^2} . \frac{3}{2^2} . \left(\frac{a'}{a}\right)^3 + \&c.\right] \cos. \omega$$

$$+ \&c.$$

and if we reject, which we may do in this case, since

$$\frac{a'}{a} = \frac{\text{rad. ☽'s orbit}}{\text{rad. ⊕'s orbit}} = \frac{1}{400}, \text{ nearly},$$

the terms involving $\left(\frac{a'}{a}\right)^2$, $\left(\frac{a'}{a}\right)^3$, &c. we shall merely have

$$\frac{1}{y} = \frac{1}{a}, \text{ and, accordingly,}$$

$$R = \frac{m' a}{a'^2} \cos.(n' t - n t + \epsilon' - \epsilon) - \frac{m'}{a}.$$

Now, in the differential dR, they are solely the ordinates x, y, of the disturbed body which are supposed to vary: consequently in the above expression $n t$ alone varies, therefore

$$2 . d R = \frac{2 m' a n}{a'^2} \sin.(n' t - n t + \epsilon' - \epsilon) d t;$$

but

$$2 \int d R = \frac{2 m' n a}{a'^2 . (n - n')} \cos.(n' t - n t + \epsilon' - \epsilon) + \text{corr.}$$

which correction we may suppose equal to $\frac{2 k}{a}$.

Again, since $\frac{r d R}{d r} = \frac{a . dR}{d a}$, we have

$$r . \frac{dR}{dr} = \frac{m'}{a} + \frac{m' a}{a'^2} \cos.(n' t - n t + \epsilon' - \epsilon),$$

and, accordingly,

$$2 \int d R + r . \frac{d R}{dr} =$$

$$\frac{m'}{a} + \frac{2 k}{a} + \frac{m' a}{a'^2} \left(1 + \frac{2 n}{n - n'}\right) \cos.(n' t - n t + \epsilon' - \epsilon).$$

If we now divide every term of the equation of p. 258. by a^2, and, μ being supposed $= 1$, write n^2 instead of $\frac{1}{a^3}$, there will result

$$O = \frac{1}{a^2} . \frac{d^2 . (r \delta r)}{d t^2} + \frac{N^2}{a^2} . r \delta r + m' n^2 + 2 k n^2 .$$

$$+ \frac{m'}{a a'^2} \left(1 + \frac{2 n}{n - n'}\right) \cos.(n' t - n t + \epsilon' - \epsilon).$$

Now this equation, as it was stated in p. 262. is precisely under the conditions requisite for that peculiar method of integration which was described in pages 97, 98, &c.: and, according to that method,

$$\frac{r\,\delta r}{a^2} = -\frac{n^2}{N^2}(m' + 2k) +$$

$$\frac{m'}{aa'^2}\cdot\frac{1}{(n-n')^2 - N^2}\left(1 + \frac{2n}{n-n'}\right)\cos.\,(n't - nt + \epsilon' - \epsilon),$$

which is a result independent of the eccentricity of the orbit.

The last term in the preceding expression is periodic. On the first depends the alteration produced in the constant part of the radius by the disturbing force: let δa represent that alteration, then

$$\frac{\delta a}{a}\left(= \frac{r\,\delta r}{a^2} = \frac{a\,\delta a}{a^2}\right) = -\frac{n^2}{N^2}(m' + 2k);$$

k being an indeterminate quantity. The coefficient of the last term may immediately be computed: for since $\frac{n}{n'} = .0748013$,

$$1 + \frac{2n}{n-n'} = \frac{775596}{925198};$$

$$\text{but } \frac{a'}{a} = \frac{27.2}{10661}\left(= \frac{1}{400}, \text{ nearly,}\right)$$

and (m representing the Earth's mass),

$$\frac{m'}{m+m'} = \frac{1}{59.6},$$

a result taken from the theory of the tides. Moreover

$$n'^2 = \frac{m+m'}{a'^3},$$

$$\text{and } \therefore \frac{m'}{a\,.\,a'^2} = \frac{m'}{a'^3}\cdot\frac{a'}{a} = \frac{m'\,n'^2}{m+m'}\cdot\frac{a'}{a},$$

and lastly, since N^2 nearly $= n^2$,

$$\frac{n'^2}{(n-n')^2 - N^2} = \frac{n'^2}{n'^2 - 2nn'} = \frac{1}{1 - \frac{2n}{n'}},$$

computing from these data, we have the periodical part of δr, namely,

$$\frac{m' a}{a'^2 . [(n-n')^2 - N^2]} \left(1 + \frac{2n}{n-n'}\right) \cos. (n't - nt + \epsilon' - \epsilon),$$

$$= a \times .000042199, \quad \cos. (n't - nt + \epsilon' - \epsilon)\,^*,$$

The coefficient of cos. $(v' - v)$, or, since the eccentricity is neglected, of cos. $(n't - nt + \epsilon' - \epsilon)$, is, by p. 85., .0000428, which latter result was deduced from the doctrine of the centre of gravity: so that we have, very nearly agreeing, two results obtained by different methods and both methods of approximation.

We will now deduce a general expression for the inequality in longitude.

If we eliminate dr^2 from the two expressions,

$$\frac{r^2 dv^2 + dr^2}{dt^2} - \frac{2\mu}{r} + \frac{\mu}{a} + 2\int dR = 0 \ldots\ldots\ldots\ldots [1],$$

$$\frac{1}{2}\ \frac{d^2 r^2}{dt^2} - \frac{\mu}{r} + \frac{\mu}{a} + 2\int dR + r\frac{dR}{dr} = 0 \ldots\ldots\ldots [2],$$

there will result

$$\frac{r^2 dv^2}{dt^2} = \frac{r\, d^2 r}{dt^2} + \frac{\mu}{r} + \frac{r\, dR}{dr} \ldots\ldots\ldots\ldots\ldots\ldots\ldots [3].$$

---

[*] The computation is thus effected:

log. 8504 $\left(= 1 - \frac{2n}{n'}\right)$ = 3.92962
log. 9252 $\left(= 1 - \frac{n}{n'}\right)$ = 3.96623
log. 59.6 $\left(= \frac{m+m'}{m'}\right)$ = 1.77524
log. 10661.................. = 4.02779
13.69888

log. 10^4............... 4
log. 7756 .... 3.88963
log. 27.2 .... 1.43456
9.32419
13.69888
5.62531
and the corresponding number is .00004219.

Laplace's coefficient (see *Mec. Cel.* tom. II. p. 108.) is −.000042808; but his argument is $U - v$, in which U is the Moon's longitude seen from the centre of the Earth, and v the Earth's longitude seen from the centre of the Sun; $n't$ or v', therefore, in the expression of the text, corresponds to $180 + U$, and, accordingly, cos. $(n't - nt + \epsilon' - \epsilon)$, or cos. $(v' - v)$ corresponds to cos. $(180 + U - v)$, or − cos. $(U - v)$.

Now R depends on the disturbing force, and, if that force be abstracted, we have

$$\frac{r^2\, d v^2}{d t^2} = \frac{r\, d^2 r}{d t^2} + \frac{\mu}{r} \ldots\ldots\ldots\ldots\ldots [4],$$

in which, as it is evident, the values of r, v, $d r$, $d v$, are *elliptical* values. When, therefore, the disturbing force acts, the terms r, v, $d v$, $d r$ may be feigned to consist of two parts, r and δr, v and δv, $d v$ and $\delta d v$, $d r$ and $\delta d r$, respectively: the first part in each being the quantity's elliptical value, the second arising entirely from the disturbing force, and involving, since R involves it, the mass of the disturbing body. Substitute in the equation [3], these augmented values of r, v, &c.: then, by virtue of the equation [4], and by the neglect of terms involving $(\delta v)^2$, $(\delta r)^2$, $(m')^2$, $\delta r \cdot \frac{d R}{d r}$, &c. the equation after reduction, will be

$$\frac{2\, r^2\, d v \,.\, d\, \delta v}{d t^2} + \frac{2\, r\, \delta r \,.\, d v^2}{d t^2} =$$

$$\frac{r\, d^2 \delta r + \delta r \,.\, d^2 r}{d t^2} + \frac{\mu\, \delta r}{r^2} + r \,.\, \frac{d R}{d r},$$

or, substituting, instead of $d v^2$, that value of it which is contained in the equation [4],

$$\frac{2 r^2 \,.\, d v \,.\, d\, \delta v}{d t^2} = \frac{r\, d^2 \delta r - \delta r \,.\, d^2 r}{d t^2} - \frac{3\, \mu\, r\, \delta r}{r^3} + r \,.\, \frac{d R}{d r}\,.$$

In this last equation substitute, instead of $\frac{\mu\, r\, \delta r}{r^3}$, that its value, which, according to the method just described (l. 10, &c.) may be derived from the equation [2], and then there will result

$$\frac{2\, r^2 \,.\, d v \,.\, d\, \delta v}{d t^2} = \frac{4\, r\, d^2 \delta r + 2\, d^2 r \,.\, \delta r + 6\, d r\, d\, \delta r}{d t^2}$$

$$+\, 4\, r \,.\, \frac{d R}{d r} + 6 \int d R.$$

Now the first term of the right-hand side of the equation equals

$$2.\ \frac{d(dr.\delta r + 2r\,d\delta r)}{dt^2},$$

and, consequently,

$$d\delta v = \frac{d(dr\delta r + 2r\,d\delta r) + dt^2\left(3\int dR + 2r.\frac{dR}{dr} + \frac{dR}{dr}\delta r\right)}{r^2.dv},$$

now the last term, $\frac{dR}{dr}\delta r$, may be rejected: for, the process is founded, partly, on the rejection of terms involving the square of the disturbing force: but R, and consequently $\frac{dR}{dr}$, contains m', so does δr; therefore, $\frac{dR}{dr}\delta r$ contains $(m')^2$. Rejecting it, then, and integrating the resulting expression,

$$\delta v = \frac{dr.\delta r + 2r.d\delta r}{r^2\,dv} + \frac{dt^2}{r^2\,dv}\int\!\!\int 3\,dR + \frac{2}{r^2\,dv}\int r\,\frac{dR}{dr}$$

$$= \frac{2\,d(r\delta r) - dr.\delta r}{a^2.n\,dt} + \frac{3a}{\mu}\int\!\!\int n\,dt.dR + \frac{2a}{\mu}\int n\,dt.r.\frac{dR}{dr},$$

since $dt = \frac{r^2\,dv}{h} = \frac{r^2\,dv}{\sqrt{\mu a}}$ (neglecting e^2)

$$= \frac{r^2\,dv}{\sqrt{\mu}.a^{\frac{3}{2}}} \times a = \frac{r^2\,dv\,n\,a}{\mu}.$$

We have now obtained a general expression for δv; or, at least, an expression which is so, on the assumed rejection of the squares of the disturbing force, the eccentricity, &c. If a more explicit value of δv be required, it will be necessary to substitute for R, the series (see p. 262.) which represents its value; or, if we adhere to the present subject of investigation (which is the Earth's perturbation by the action of the Moon) by merely substituting for R the value (see p. 264.),

$$\frac{m'a}{a'^2}\cos.(n't - nt + \epsilon' - \epsilon) - \frac{m'}{a}.$$

Hence,

$$\frac{3a}{\mu}\iint ndt\,.\,dR = 3knt - \frac{3m'}{\mu}\,.\left(\frac{n}{n-n'}\right)^2\frac{a^2}{a'^2}\,\text{sin.}\,(n't - nt + \epsilon' - \epsilon),$$

(k being an arbitrary quantity introduced for the purposes of correction). Again,

$$2a\int r\,.\,\frac{dR}{dr}\,.\,n\,dt = \frac{m'}{\mu}\left(2\,nt - \frac{2\,n}{n-n'}\,.\,\frac{a^2}{a'^2}\,\text{sin.}\,(n't - n\,t + \epsilon' - \epsilon)\right),$$

and

$$\frac{2\,.\,d(r\delta r)}{a^2\,n\,dt} = \frac{2m'}{\mu\,.\,n}\,.\,\frac{a^2}{a'^2}\left(\frac{n^2}{(n-n')^2 - N^2}\right)(3n - n')\,\text{sin.}\,(n't - n\,t + \epsilon' - \epsilon).$$

If we substitute these values in the expression for $\delta\,v$, and then reduce the expression by the ordinary methods, there will result

$$\delta\,v =$$

$$3\,k\,n\,t + 2\,m'\,.n\,t +$$

$$\frac{m'n}{n-n'}\left\{\begin{array}{l}\dfrac{n}{n-n'}\,.\,\dfrac{a^2}{a'^2}\\ +\dfrac{2\,N^2}{(n-n')^2 - N^2}\left(\dfrac{3n-n'}{n-n'}\,.\,\dfrac{a^2}{a'^2}\right)\end{array}\right\}\text{sin.}\,(n'\,t - n\,t + \epsilon' - \epsilon).$$

$\delta\,v$ is, in this expression, the variation or inequality in longitude arising from the disturbing force: but $n\,t$ expounds the mean motion: no term then of $n\,t$ can possibly enter into the expression for $\delta\,v$: and accordingly, we must have

$$3\,k\,n\,t + 2\,m'n\,t = 0\,;$$

$$\text{whence,}\quad k = -\,\frac{2\,m'}{3}\,.$$

If, which is nearly the case, we make $N^2 = n^2$, and besides write m instead of $\frac{n}{n'}$, (see p. 140.)

$$\frac{1}{59\,.6}\ \text{for}\ \frac{m'}{a'^3\,n'^2}\left(= \frac{☽\text{'s mass}}{☽\text{'s mass} + ⊕\text{'s mass}}\right),$$

we shall have

$$\delta v = \frac{1}{59.6 \, . \, m^2} \cdot \frac{a'}{a} \cdot \frac{m}{1-m} \times$$

$$\left[\frac{m}{1-m} - \frac{2m^2}{1-m}\left(\frac{1-3m}{1-m}\right)\right] \sin.(n't - nt + \epsilon' - \epsilon),$$

and, if according to the numerical values of a, a', m, as given in p. 265. we compute * the coefficient of the preceding term, we shall have

---

Computation.

* $m = .0748013$
$a' = 27.2$
$a = 106691$

1st, $\dfrac{2m^2}{1-2m} \cdot \dfrac{1-3m}{1-m}$, computed.

log. $(1-3m)$ ...... $\bar{1}.88963$ .... log. $(1-m)$ .... $\bar{1}.96623$
log. $2m^2$...... $\bar{2}.04883$ ....log. $(1-2m)$...... .. $\bar{1}.92972$

$\bar{3}.93846$ (a) $\bar{1}.89595$
$\bar{1}.89595$ (a)

$\bar{2}.04251$......No. = .011028 (n).

2dly, $\dfrac{m}{1-m}$ computed.

log. m.........$\bar{2}.87391$
log, $(1-m)$...$\bar{1}.96623$

$\bar{2}.90767$ ... No. = .08084
(n) = .011028

$\therefore \dfrac{m}{1-m} - \dfrac{2m^2}{1-2m} \cdot \dfrac{1-3m}{1-m} = .06982$, and log. = $\bar{2}.84397$

3dly, $59.6\,am\,.(1-m)$ computed

log. m...............$\bar{2}.87391$
log. a...............4.02779
log. 59.6 ....1.77524
log. $(1-m)$.........$\bar{1}.96623$

4.64317 (b)

log. a' = 1.43457
log. arc = rad. 1.75812

2.03666
4.64317

$\bar{3}.39349$
and the No. is .002475
or, in seconds, 8′.9

$$\delta v = 8''.9 . \sin. (n't - nt + \epsilon' - \epsilon),$$

$$\text{or}, = 8''.9 . \sin. (v' - v).$$

Since, the eccentricity being supposed nothing, the mean and true motions are the same.

The above value of δv agrees with that which, derived from different principles, was given in p. 85.

If, in the preceding expression, we write, as in p. $180^0 + U$ instead of v', we shall have

$$\delta v = 8''.9 . \sin. (180^0 + U - v)$$
$$= - 8''.9 . \sin. (U - v),$$

which, very nearly, is Laplace's expression: (see *Mec. Cel.* tom. III. p. 108.

The perturbations then in the Solar parallax and longitude are, after the establishment of the equations of pp. 258. 268., very easily deduced. One cause of the facility of deduction is the abstraction of the condition of the eccentricity, which abstraction is arbitrary or hypothetical: another, and which must always exist, is the minuteness of the radius of the Lunar Orbit, compared with that of the Solar: a minuteness such as to render unnecessary, as a compendium of computation, that formula (see p. 262.) by which R is expressed in a series of terms involving the cosines of multiple arcs.

The deduction, in the present Chapter, of the perturbations of the Solar Orbit by the Moon's action is intended principally to illustrate the use of the newly derived differential equations: but the results serve, besides, to confirm, or are confirmed by, those results, which, in Chap. VI. were obtained by the method * of the centre of gravity: a method, (if we look to its use in the Theory of Perturbations) partial, and restricted, almost completely, to the case to which it was applied.

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* This method of determining the perturbation of a primary by the action of its satellite (for such is the case of the Moon disturbing the Earth's motion) originated with Dalembert (see *Recherches sur differens points dans le systême du monde.* tom. II. pp. 20. 47, &c.). In the same treatise, however, that acute writer shews that we ought to prefer, in investigating the perturbation of the planets, a systematic integration of the differential equations: or, in other words, a direct solution of the Problem of the Three Bodies.

The uses of the differential equations of pp. 258. 268. are not sufficiently illustrated by the preceding case. They will be more adequately illustrated by the research of the Earth's perturbations from the action of Jupiter, and especially, if we retain in it, the condition of the eccentricity of the Solar Orbit. This case will serve too, more fully than the preceding, to shew the utility of developing *R* into a series of terms involving the cosines of multiple arcs: and, will, accordingly, illustrate one ground of distinction (see p. 254.) between the Lunar and Planetary Theories. But it will not serve as a characteristic illustration of this latter point.

CHAP. XVII.

On the Development of R *in terms of the Cosines of the Mean Motions of the disturbed and disturbing Planets. On the Method of Computing the Coefficients of the Development, when the Radius of the Orbit of the Disturbed Body differs considerably from that of the Disturbing: Application of the Formulæ to the Case of Jupiter disturbing the Earth. New Formulæ necessary when the Radii of the Orbits of the two Bodies are nearly Equal.*

If we revert to p. 66, we shall find that, when the inclination of the orbits of the disturbing and disturbed bodies is neglected,

$$R = \frac{m' r}{r'^2} \cdot \cos.(v' - v) - \frac{m'}{\sqrt{[r'^2 - 2rr' \cos.(v'-v) + r^2]}} \cdot$$

In the instance given in the preceding Chapter, great facility was afforded to the computation, by assuming the orbits, both of the revolving and of the disturbing body, devoid of eccentricity. In consequence of which assumption, we had

$$r = a, \qquad r' = a',$$

$$v = nt + \epsilon, \quad v' = n't + \epsilon'.$$

This was one source of facility: another (and, in an Elementary Treatise, we cannot well insist too much on the important points) was the minuteness of $\frac{r'}{r}$. By reason of that minuteness the series for $\frac{1}{y}$ (see p. 261.) converged so rapidly, that it was sufficiently exact to represent R (see p. 263.) simply by

$$\frac{m' a}{a'^2} \cos. (n' t - n t + \epsilon' - \epsilon) - \frac{m'}{a}.$$

There will then be, almost in every case of planetary disturbance, (for scarcely any case is equally simple with the preceding) two causes of change in the value of R: namely, the eccentricities of the orbits, and the slower convergency of the series representing $\frac{1}{y}$, by reason of the *less* minuteness of $\frac{r'}{r}$, or $\frac{r}{r'}$. By the effect of this latter, $\frac{1}{m'} R$, instead of the foregoing simple expression, would be represented by a series such as

$$\frac{1}{2} \mathrm{A} + \mathrm{B} . \cos. \omega + \Gamma . \cos. 2 \omega + \&c.$$

in which, if we abstract the eccentricities, A, B, Γ, &c. would solely involve, or be *functions* of, a, a', and ω would be

$$n' t - n t + \epsilon' - \epsilon.$$

If the eccentricities be taken account of, and be represented by e, e', we have, for estimating A, B, Γ, &c. and $\cos. \omega$, $\cos. 2 \omega$, $\cos. 3 \omega$, &c. the following equations (see pp. 31, 32.)

$$r = a \left(1 - e . \cos. U + \frac{e^2}{2} - \frac{e^2}{2} \cos. 2 U + \&c.\right)$$

$$r' = a' \left(1 - e' . \cos. U' + \frac{e'^2}{2} - \frac{e'^2}{2} \cos. 2 U' + \&c.\right)$$

$$v = n t + \epsilon + 2 e \sin. U + \frac{5 e^2}{4} \sin. 2 U + \&c.$$

$$v' = n' t + \epsilon' + 2 e' \sin. U' + \frac{5 e'^2}{4} \sin. 2 U' + \&c.$$

U, U' being respectively equal to $n t + \epsilon - \pi$, and $n' t + \epsilon' - \pi'$.

Since $\cos. \omega$, $\cos. 2 \omega$, $\cos. 3 \omega$, &c. are required, it will be most convenient to deduce at once a general expression for $\cos. p \omega$, or $\cos. p (v' - v)$, from the values of v' and v, that have just been given

$$\cos.\, p\,(v' - v) = {}^{*}$$

$$\begin{aligned}
&\cos.\,(pn'l - pnt + p\epsilon' - p\epsilon)\,[1 - p^2\,.\,(e^2 + e'^2)] \\
&+ pe\,.\cos.\,(pn't - pnt + p\epsilon' - p\epsilon - nt - \epsilon + \pi) \\
&- pe\,.\cos.\,(pn't - pnt + p\epsilon' - p\epsilon + nt + \epsilon - \pi) \\
&+ pe'\,.\cos.\,(pn't - pnt + p\epsilon' - p\epsilon + n't + \epsilon' - \pi') \\
&- pe'\,.\cos.\,(pn't - pnt + p\epsilon' - p\epsilon - n't - \epsilon' + \pi')
\end{aligned}$$

$$+ p^2 e e' \left\{ \begin{array}{l}
\cos.\,(pn't - pnt + p\epsilon' - p\epsilon - n't + nt - \epsilon' + \epsilon + \pi' - \pi) \\
+ \cos.\,(pn't - pnt + p\epsilon' - p\epsilon + n't - nt + \epsilon' - \epsilon - \pi' + \pi) \\
- \cos.\,(pn't - pnt + p\epsilon' - p\epsilon + n't + nt + \epsilon' + \epsilon - \pi' - \pi) \\
- \cos.\,(pn't - pnt + p\epsilon' - p\epsilon - n't - nt - \epsilon' - \epsilon + \pi' + \pi)
\end{array} \right\}$$

$$+ p\frac{e^2}{2} \left\{ \begin{array}{l}
\left(p + \frac{5}{2}\right).\cos.\,(p'nt - pnt + p\epsilon' - p\epsilon - 2nt - 2\epsilon + 2\pi) \\
+ \left(p - \frac{5}{2}\right).\cos.\,(pn't - pnt + p\epsilon' - p\epsilon + 2nt + 2\epsilon - 2\pi)
\end{array} \right\}$$

$$+ p\frac{e'^2}{2} \left\{ \begin{array}{l}
\left(p + \frac{5}{2}\right).\cos.\,(pn't - pnt + p\epsilon' - p\epsilon + 2n't + 2\epsilon' - 2\pi') \\
+ \left(p - \frac{5}{2}\right).\cos.\,(pn't - pnt + p\epsilon' - p\epsilon - 2n't - 2\epsilon' + 2\pi')
\end{array} \right\}$$

We must now attend to the other point, namely, the values of P, P', P'', &c. in the series

$$\frac{1}{2} P + P'\,.\cos.\,\omega + P''\,.\cos.\,2\,\omega + \text{\&c.}$$

supposing that to be the development of $\dfrac{1}{\sqrt{(r'^2 - 2\,r\,r' \cos.\,\omega + r^2)}}$.

If there were no eccentricity, or if r, r' were equal to a, a', then, the coefficients of the terms of the preceding series would (as it was explained in p. 274.) be functions of a, a', and their values would be assignable by the formulæ of p. 260: but the orbits being eccentric, a, a', become r, r', or, $a - a\,e\,.\cos.\,U + \&c.$ $a' - a'e'\cos.\,U' + \&c.$ and, accordingly, we may conceive a, a', to become $a + \Delta\,a$, $a' + \Delta\,a'$,

---

* In deducing the value of $\cos.\, p\,\omega$, since account is made of terms involving e^2, e'^2, &c. we must write for $\cos.\,(2\,e\,.\,\sin.\,z)$ (z being supposed to be an arc) not $2\,e \sin z$, as in pp. 103. 104. of *Trigonometry*; but $1 - \dfrac{(2\,e\,.\,\sin.\,z)^2}{2} = 1 - e^2 + e^2 \cos.\, 2\,z.$

$$\Delta\,a \text{ being } = -\,a\left(e\cos.\,U - \frac{e^2}{2} + \frac{e^2}{2}\cos.\,2\,U\right),$$

$$\Delta\,a' \quad = -\,a'\left(e'\cos.\,U' - \frac{e'^2}{2} + \frac{e'^2}{2}\cos.\,2\,U\right),$$

and in such a case, we shall have, (see *Principles of Anal. Calc.* pp. 86, &c.), supposing that

$$\frac{1}{\surd\,(a'^2 - 2\,a\,a'\cos.\,\omega + a^2)} = \frac{1}{2}\,A + B\,.\cos.\,2\omega + C\,.\cos.\,3\,\omega + \&c.$$

$$P = A + \frac{d\,A}{da}\Delta\,a + \frac{d\,A}{da'}\Delta\,a' + \frac{d^2\,A}{1\,.\,2\,.\,d\,a^2}(\Delta\,a)^2, + \&c.$$

$$P' = B + \frac{d\,B}{d\,a}\Delta\,a + \frac{dB}{d\,a'}\Delta\,a' + \frac{d^2\,B}{1\,.\,2\,.\,da^2}(\Delta\,a)^2 + \&c.$$

$$P'' = C + \&c.$$

These expressions will be much abridged if we neglect, which we may do in most of the planetary theories, the terms that involve the squares and cubes of e, e'. If such terms be neglected,

$$P = A - a\frac{d\,A}{d\,a}\,e\,.\cos.\,U - a'\frac{d\,A}{d\,a'}\,e'\,.\cos.\,U',$$

$$P' = B - a\frac{d\,B}{d\,a}\,e\,.\cos.\,U - a'\frac{d\,B}{d\,a'}\,e'\,.\cos.\,U',$$

$$P'' = C - a\frac{d\,C}{d\,a}\,e\,.\cos.\,U - a'\frac{d\,C}{d\,a'}\,e'\,.\cos.\,U',$$

$$P''' = D - a\frac{dD}{d\,a}\,e\,.\cos.\,U - a'\frac{d\,D}{d\,a'}\,e'\,.\cos.\,U'.$$

The values of A, B, C, D, are to be determined by means of the formulæ of p. 260. Now $\frac{a}{a'} = \alpha$ being a fraction, the quantities A, B, C, D, successively decrease: in the case of Venus disturbed by Jupiter, since $\alpha = .13907116$, they would so rapidly decrease that it would be sufficient to retain and compute three, namely, A, B, C: and, it is enough for exactness, if when Jupiter disturbs the Earth, and $\alpha = .19226461$, we make account of four or five: and Clairaut, who first computed this latter case, has not much farther extended his computation: see *Mem. Acad. Paris*, 1754.

But, if D be very small, $\frac{dD}{da} e . \cos. U$, $\frac{dD}{da'} e' . \cos. U$ will be much smaller, by reason of the minuteness of e, e': and, in the case just adverted to, that of the Earth's perturbations by Jupiter, they may be neglected; or, with sufficient exactness, we may make $P''' = D$.

Let us take this case, and neglect the terms that involve the e^2, e'^2, e^3, &c. there will then result,

$$\frac{1}{\sqrt{(r'^2 - 2rr' . \cos. \omega + r^2)}} =$$

$$\frac{1}{2} A + B . \cos. (n't - nt + \epsilon' - \epsilon) + C . \cos. 2 (n't - nt + \epsilon' - \epsilon),$$

$$+ D . \cos. 3 (n't - nt + \epsilon' - \epsilon)$$

$$- \frac{a}{2} . \frac{dA}{da} e . \cos. (nt + \epsilon - \pi) - \frac{a'}{2} . \frac{dA}{da'} e' . \cos. (n't + \epsilon' - \pi')$$

$$+ \left(B - \frac{a}{2} . \frac{dB}{da}\right) e . \cos. (n't - 2nt + \epsilon' - 2\epsilon + \pi)$$

$$- \left(B + \frac{a}{2} . \frac{dB}{da}\right) e . \cos. (n't + \epsilon' - \pi)$$

$$+ \left(B - \frac{a'}{2} . \frac{dB}{da'}\right) e' . \cos. (nt - 2n't + \epsilon - 2\epsilon' + \pi')$$

$$- \left(B + \frac{a'}{2} . \frac{dB}{da'}\right) e' . \cos. (nt + \epsilon - \pi')$$

$$+ \left(2C - \frac{a}{2} \frac{dC}{da}\right) e \cos. (2n't - 3nt + 2\epsilon' - 3\epsilon + \pi)$$

$$- \left(2C + \frac{a}{2} \frac{dC}{da}\right) e . \cos. (2n't - nt + 2\epsilon' - \epsilon - \pi)$$

$$+ \left(2C - \frac{a'}{2} \frac{dC}{da'}\right) e' . \cos. (3n't - 2nt + 3\epsilon' - 2\epsilon - \pi')$$

$$- \left(2C + \frac{a'}{2} \frac{dC}{da'}\right) e' . \cos. (n't - 2nt + \epsilon' - 2\epsilon + \pi').$$

the succeeding terms involving the products $D\,e$, $D\,e'$, may, from their minuteness, be neglected.

But in order to find the value of R, the value of the last trinomial (multiplied into m') must be (see p. 273.) subtracted from $\frac{m'r}{r'^2}\cos.\,(v'-v)$.

Now,

$$\frac{m'r}{r'^2}\cos.\,(v'-v)$$

$$=\text{(nearly) }\frac{m'a}{a'^2}\cdot\frac{1-e\,.\cos.\,(n\,t+\epsilon-\pi)}{1-2e'\cos.\,(n'\,t+\epsilon'-\pi')}\,.\cos.\,(v'-v)$$

$$=\frac{m'a}{a'^2}\,[1-e\,.\cos.\,(nt+\epsilon-\pi)+2e'\,.\cos.\,(n't+\epsilon'-\pi')]\cos.\,(v'-v)$$

$$=\frac{m'a}{a'^2}\,.\cos.\,(v'-v)$$

$$-\frac{m'a}{2\,a'^2}\,e\,.\cos.\,(n'\,t-2\,n\,t+\epsilon'-2\,\epsilon+\pi),$$

$$-\frac{m'a}{2\,a'^2}\,e\,.\cos.\,(n'\,t+\epsilon'-\pi)$$

$$+\frac{m'a}{a'^2}\,e'\,.\cos.\,(n\,t-2\,n'\,t+\epsilon-2\,\epsilon'+\pi')$$

$$+\frac{m'a}{a'^2}\,e'\,.\cos.\,(n\,t+\epsilon-\pi')$$

$=$ (see p. 275.)

$$\frac{m'}{a'^2}\,.\cos.\,(n'\,t-n\,t+\epsilon'-\epsilon)$$

$$+\frac{m'a}{2\,a'^2}\,e\,.\cos.\,(n'\,t-2\,n\,t+\epsilon'-2\,\epsilon+\pi)$$

$$-\frac{3\,m'a}{2\,a'^2}\,e\,.\cos.\,(n'\,t+\epsilon'-\pi)$$

$$+\frac{2\,m'a}{a'^2}\,e'\,.\cos.\,(n\,t-2\,n'\,t+\epsilon-2\epsilon'+\pi),$$

the coefficient of the other term which involves e' being nothing.

If we now combine this result with the preceding, we shall have

$$\begin{aligned}
R = & -\frac{m'A}{2} + m'\left(\frac{a}{a'^2} - B\right)\cos.(n't - nt + \epsilon' - \epsilon) \\
& - m'C \,.\, \cos.(2n't - 2nt + 2\epsilon' - 2\epsilon) \\
& - m'D \,.\, \cos.(3n't - 3nt + 3\epsilon' - 3\epsilon) \\
& + \frac{m'}{2}\, a\,\frac{dA}{da}\, e \,.\, \cos.(nt + \epsilon - \pi) + \frac{m'}{2}\, a'\frac{dA}{da'}\, e' \,.\, \cos.(n't + \epsilon' - \pi') \\
& + \frac{m'}{2}\left(\frac{a}{a'^2} - 2B + a\frac{dB}{da}\right) e \,.\, \cos.(n't - 2nt + \epsilon' - 2\epsilon + \pi) \\
& - \frac{m'}{2}\left(\frac{3a}{a'^2} - 2B - a\frac{dB}{da}\right) e \,.\, \cos.(n't + \epsilon' - \pi) \\
& + \frac{m'}{2}\left(\frac{4a}{a'^2} - 2B + a'\frac{dB}{da'}\right) e' \,.\, \cos.(nt - 2n't + \epsilon - 2\epsilon' + \pi') \\
& + \frac{m'}{2}\left(2B + a'\frac{dB}{da'}\right) e' \,.\, \cos.(nt + \epsilon - \pi') \\
& - \frac{m'}{2}\left(4C - a\frac{dC}{da}\right) e \,.\, \cos.(2n't - 3nt + 2\epsilon' - 3\epsilon + \pi) \\
& + \frac{m'}{2}\left(4C + a\frac{dC}{da}\right) e \,.\, \cos.(2n't - nt + 2\epsilon' - \epsilon - \pi) \\
& - \frac{m'}{2}\left(4C - a'\frac{dC}{da'}\right) e' \,.\, \cos.(3n't - 2nt + 3\epsilon' - 2\epsilon - \pi') \\
& + \frac{m'}{2}\left(4C + a'\frac{dC}{da'}\right) e' \,.\, \cos.(n't - 2nt + \epsilon' - 2\epsilon + \pi'). \\
& + \&c.
\end{aligned}$$

The above is the developed expression for R, and such as must be used, when, in a specific instance, it is necessary to compute, arithmetically, the coefficients of the equations. But, if we make certain conventions, R may be much more abridgedly expressed: for instance,

$$p \,.\, (n't - nt + \epsilon' - \epsilon),$$

is a general expression for all the terms composing the *Variation* (see pp. 218, 219.): and

$$\begin{aligned}
& p \,.\, (n't - nt + \epsilon' - \epsilon) + nt + \epsilon - \pi, \\
& p \,.\, (n't - nt + \epsilon' - \epsilon) + nt + \epsilon - \pi', \\
\text{or } & p \,.\, (n't - nt + \epsilon' - \epsilon) + n't + \epsilon' - \pi',
\end{aligned}$$

are two general expressions, under which all the other arguments are comprehended, p being intended to designate the numbers $0, 1, -1, 2, -2$, &c.

The coefficients may also be generally expressed: for instance,

$$\frac{m'}{2}\left(2pC + a \,.\, \frac{dC}{da}\right),$$

on making $p = -2$, represents the coefficient of the term in the ninth line of the preceding value of R: and the same formula, on making $p = 2$, represents the coefficient in the tenth line: but

$$p \,.\, (n't - nt + \epsilon' - \epsilon) + nt + \epsilon - \pi,$$

on making $p = -2$ and 2, represent also respectively the arguments in the ninth and tenth lines, therefore

$$\frac{m'}{2}\left(2pC + a \,.\, \frac{dC}{da}\right) e \cos. \left[p(n't - nt + \epsilon' - \epsilon) + nt + \epsilon - \pi\right),$$

(p standing for -2 and 2) will represent the terms in the ninth and tenth lines.

Again, if we except the parts $\frac{m'a}{2a'^2}$, $-\frac{3m'a}{2a'^2}$ in the fifth and sixth lines,

$$\frac{m'}{2}\left(2pB + a \,.\, \frac{dB}{da}\right) e \cos. \left[p(n't - nt + \epsilon' - \epsilon) + nt + \epsilon - \pi\right],$$

(p representing -1 and 1), will represent the terms in the fifth and sixth lines

$$\frac{m'}{2}\left(2pA + a \,.\, \frac{dA}{da}\right) \cos. p\left[(n't - nt + \epsilon' - \epsilon) + nt + \epsilon - \pi\right],$$

making $p = 0$, will represent the first term in the fourth line: Hence, generally, if $A^{(p)}$ represents the coefficient of $\cos. p\,\omega$, the terms of which we have spoken will be all comprehended in the expression,

$$\frac{m'}{2}\left(2pA^{(p)} + a \,.\, \frac{dA^{(p)}}{da}\right) e \,.\cos. \left[p(n't - nt + \epsilon' - \epsilon) + nt + \epsilon - \pi\right],$$

and, in like manner, it may be made easily to appear that

$$\frac{m'}{2}\left(a' . \frac{d A^{(p)}}{d a'} - 2pA^{(p)}\right) e' . \cos. [p .(n't - nt + \epsilon' - \epsilon) + n't + \epsilon' - \pi'),$$

is the general representation of the terms in the seventh, eighth, eleventh, twelfth lines, and of the second term in the fourth line of the preceding value of R.

R, therefore, by virtue of the preceding conventions, &c. may be thus more concisely expressed :

$$\begin{aligned}
{}^{*}R = &- \frac{m'}{2} \Sigma A^{(p)} . \cos. p\,(n't - nt + \epsilon' - \epsilon) \\
&+ \frac{m'a}{a'^2} . \cos. (n't - nt + \epsilon' - \epsilon) \\
&+ \frac{m'a}{2a'^2} e . \cos. (n't - 2nt + \epsilon' - 2\epsilon + \pi) \\
&- \frac{3m'a}{2a'^2} e . \cos. (n't + \epsilon' - \pi) \\
&+ \frac{4m'a}{2a'^2} e' . \cos. (nt - 2n't + \epsilon - 2\epsilon' + \pi') \\
+ \frac{m'}{2} \Sigma \left(2pA^{(p)} + a . \frac{d A^{(p)}}{d a}\right) &e . \cos. [p\,(n't - nt + \epsilon' - \epsilon) + nt + \epsilon - \pi] \\
- \frac{m'}{2} \Sigma \left(2pA^{(p)} - a' . \frac{d A^{(p)}}{d a'}\right) &e' . \cos. [p(n't - nt + \epsilon' - \epsilon) + n't + \epsilon' - \pi']
\end{aligned}$$

the symbol Σ (significant of summation) denoting that all the values of the terms it affects are to be taken, such values arising on writing 0, 1, − 1, 2, − 2, &c. for p.

It is convenient to use this general expression, when analytical operations are performed on R : for instance, in deducing

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* This value of R, will, on examination (an operation in the present case not very easy) be found to be the same, in effect, as Laplace's, although less abridgedly expressed : but the advantage which that great mathematician has procured, is balanced by a *defect of generality* in the expression of some of his terms. His expression, for instance, of $A^{(i)}$ in terms of $b_{\frac{1}{2}}^{(i)}$, &c. is not general : it fails when $i = 1$: and although the exception is formally made, yet it is an exception, and in the detail of computation not a little embarrassing ; see *Mec. Cel.* Liv. II. *Premiere Partie,* pp. 272. 276.

$$2\int dR + r\frac{dR}{dr},$$

which are the two last terms of the differential equation.

These terms are easily deduced, whether we use the first or second expression for R: if we take the term contained in the second line of its developed value (see p. 279.), namely,

$$-\frac{m'}{2}C.\cos.(2n't - 2nt + 2\epsilon' - 2\epsilon).$$

We shall have, with regard to such term $\left(r\frac{dR}{dr}\right.$, being equal to $\left.a.\frac{dA}{da}\right)$,

$$2\int dR + r.\frac{dR}{dr} =$$

$$-\frac{m'}{2}\left(\frac{2n}{n-n'}C + a.\frac{dC}{da}\right)\cos.(2n't - 2nt + 2\epsilon' - 2\epsilon),$$

and if we take the first term in the *less explicit* value of R, namely,

$$-\frac{m'}{2}\Sigma.A^{(p)}.\cos.p(n't - nt + \epsilon' - \epsilon),$$

(see p. 281.), there will result,

$$2\int dR + r.\frac{dR}{dr} =$$

$$-\frac{m'}{2}\Sigma\left(\frac{pn}{n-n'}.A^{(p)} + a.\frac{dA^{(p)}}{da}\right)\cos.p(n't - nt + \epsilon' - \epsilon),$$

and the corresponding terms in the value of $\frac{r\delta r}{a^2}$, after integration, will be (see pp. 97, 98.) the preceding divided respectively by

$$(2n - 2n')^2 - N^2, \text{ and } (pn - pn')^2 - N^2.$$

We will now examine a little more minutely the values of A, B, C, D, by means of which the preceding value of R is expressed: Now,

$$\frac{1}{\sqrt{(a'^2 - 2aa'\cos.\omega + a^2)}} = \tfrac{1}{2}A + B.\cos.\omega + C.\cos.2\omega + D.\cos.3\omega.$$

ω representing, in this case, $n't - nt + \epsilon' - \epsilon$: and accordingly, see p. 260.

$$\frac{1}{2}A = \frac{1}{a'} + \left(\frac{1}{2}\right)^2 \cdot \frac{a^2}{a'^3} + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \cdot \frac{a^4}{a'^5},$$

$$B = \frac{a}{a'^2}\left(1 + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{a^2}{a'^2} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{a^4}{a'^4} + \&c.\right)$$

$$C = \frac{a^2}{a'^3}\left(\frac{3}{4} + \frac{1 \cdot 3}{4} \cdot \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{a^2}{a'^2} + \&c.\right)$$

$$D = \frac{a^3}{a'^4}\left(\frac{3 \cdot 5}{4 \cdot 6} + \&c.\right)$$

Here the terms, within the brackets, successively decrease from the minuteness of $\frac{a}{a'}$, $\left(\frac{a}{a'}\right)^2$, &c. and, for the same cause, the terms themselves, B, C, D, successively become less and less.

In the Theory of the Earth's perturbation by the action of Jupiter,

$$\alpha = .19226461\,;$$

$\therefore \alpha^2 = .0369656$, $\alpha^3 = .0071071$, $\alpha^4 = .0013664$, and accordingly,

$$\frac{1}{2}A = 1.0094385,$$
$$B = .19496,$$
$$C = .02788,$$
$$D = .0045.$$

In the Theory of the perturbations of Venus by Jupiter, the terms A, B, C, D, and the terms of the series representing their values decrease more rapidly: in this case

$$\alpha = .13907116,$$

and

$$A = 2.00977, \quad B = .14008,$$
$$C = .01462, \quad D = .00169:$$

and still more rapidly than either of the preceding cases would the terms decrease, if, Venus being still the disturbed planet, Saturn should be the disturbing.

In the cases then we have just enumerated, there is no real difficulty in computing the perturbations in longitude and parallax, by means of the differential equations of pp. 258. 268. and the value of R. The series of terms expressing that value of R will not extend, by reason of the decreasing values of A, B, C, beyond a certain limit: nor will the terms of the several series that express the values of A, B, C, D. But we must now consider cases before adverted to, namely, those in which

$$\alpha = \frac{\text{radius of the orbit of the disturbed planet}}{\text{radius of the orbit of the disturbing}},$$

should not differ so much from unity, as it differed in the preceding cases. Let us suppose an extreme case, and that two of the newly discovered planets, Juno and Ceres, whose mean distances are 2.667163, 2.767406, are the mutually disturbing bodies. In such a case, it is clear, that the terms A, B, C, D, &c. would neither decrease rapidly, nor would the terms of the series that severally represent their values. The solution of the problem would, without some new device, become impracticable.

But the preceding case may be thought too unimportant to shew the *necessity* (practically speaking) of some new method of computation. It is, otherwise, however, with the Earth's perturbations by the action of Venus, which are required to be known in the construction of the Solar Tables. Now, in such a case

$$\alpha = .723323,$$

and the terms of the series of which we have spoken, in the above value of α, decrease so slowly as to be, at the least, extremely incommodious. Some new artifice is requisite for their exact summation.

The above remarks apply to the cases of the mutual perturbations of Mars and the Earth; of Jupiter and Saturn, and of Jupiter's Satellites acting on each other.

Euler, investigating the mutual perturbations of Jupiter and Saturn, perceived the failure of the ordinary methods for computing the coefficients A, B, &c. and first invented a new method. Clairaut[*], on the subject of the perturbations of the Earth by the action of Venus, invented a different method, but for the same end as Euler's. Other methods have been subsequently invented, and, Science being progressive, the last invented are better than the preceding. But of such methods there are none that are simple and obvious: and the least simple, and least obvious, but, by many degrees, the most commodious, is the one which will be described in the next Chapter. Besides its immediate and practical importance, it will illustrate the manner by which the refined and abstruse formulæ (as they are called) of Analysis, may be made subservient to the ends of Physical Science.

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[*] Lorsque l'orbite de la planéte troublante est considerablement plus grande ou plus petite que celle de la planète troublée, les series qu' expriment la distance de deux planetes et ses puissances, se presentent tout naturellement sous une forme assez convergente; mais dans les cas ou les rayons des deux orbites ont un rapport qui ne permet pas de negliger les puissances élevées, les mêmes series decroissent si peu, qu'il faut avoir recours a des *artifices particuliers* pour determiner avec precision les termes dont on a besoin. Telle est la question de l'action de Venus sur la terre, qui nous reste a traiter dans ce Memoire. Telle est aussi celle de l'action de Jupiter sur Saturne que M. Euler a consideré dans la pièce que l'Academie couronna en 1748; c'est cet habile Geomètre qui a trouvé le premier la reduction des series de l'espece dont nous avons besoin maintenant'. *Acad. des Sciences.* 1754. p. 545.

CHAP. XVIII.

On the Method of determining the Coefficients of the Development of $(r'^2 - 2rr'\cos.\omega + r^2)^{-2s}$ *when the Fraction* $\frac{r}{r'}$ *does not differ much from* 1. *Application of the Formulæ to the Mutual Perturbations of the Earth and Venus.*

THE investigation in the present Chapter, will consist of two parts: one, the deduction of the coefficients *C*, *D*, *E*, and from the two first *A* and *B*; the other, the numerical computation of *A* and *B*: this latter point will be first considered.

Make $\frac{a}{a'} = \rho$: then,

$$\frac{1}{a'\sqrt{(1 - 2\rho'.\cos.\omega + \rho'^2)}} = \frac{1}{2}A + B.\cos.\omega + C.\cos.2\omega + \&c.$$

and, multiplying each side of the equation by the differential of ω and integrating

$$\frac{1}{a'}\int\frac{d\omega}{\sqrt{(1 - 2\rho'.\cos.\omega + \rho'^2)}} = \frac{1}{2}A\omega + B\sin.\omega + \frac{1}{2}C.\sin.2\omega + \&c.$$

let π designate the semi-circumference of a circle, the radius of which is 1, and suppose the integral of the above equation to be taken within the values of $\omega = 0$, and $\omega = \pi$, then

$$\frac{a'}{2}A.\pi = \int\frac{d\omega}{\sqrt{(1 - 2\rho'.\cos.\omega + \rho'^2)}},$$

and the difficulty now (see p. 284.), under somewhat of a different shape, is to find the preceding integral.

Now,

$$1 - 2\rho' \cos.\,\omega + \rho'^2 = (1+\rho')^2 \cdot \left(1 - \frac{4\,\rho'}{(1+\rho')^2}\cos.^2\frac{\omega}{2}\right).$$

Make $\rho^2 = \frac{4\,\rho'}{(1+\rho')^2}$, or, which is the same, assume

$$\rho' = \frac{1 - \sqrt{(1-\rho^2)}}{1 + \sqrt{(1-\rho^2)}},$$

and, besides, let x represent cos. $\frac{\omega}{2}$: then

$$\frac{a'}{2} A\,\pi = \frac{2}{1+\rho'} \cdot \int \frac{d\,x}{\sqrt{(1-x^2)\,(1-\rho^2 x^2)}}.$$

Now, the differential expression on the right-hand side of the equation is such, that if we assume

$$u' = \frac{2}{1+\rho'} \cdot x\sqrt{\left(\frac{1-x^2}{1-\rho^2 x^2}\right)},$$

there will result

$$\frac{d\,x}{\sqrt{[(1-x^2)\,.\,(1-\rho^2 x^2)]}} = \frac{1+\rho'}{2} \cdot \frac{du'}{\sqrt{(1-u'^2)\,(1-\rho'^2 u'^2)}},$$

and, accordingly, if we continue to make like assumptions, viz.

$$\rho'' = \frac{1-\sqrt{(1-\rho'^2)}}{1+\sqrt{(1-\rho'^2)}}, \quad u'' = \frac{2}{1+\rho''} \cdot \frac{u'\sqrt{(1-u'^2)}}{\sqrt{(1-\rho'^2 u'^2)}},$$

$$\rho''' = \frac{1-\sqrt{(1-\rho''^2)}}{1+\sqrt{(1-\rho''^2)}}, \quad u''' = \frac{2}{1+\rho'''} \cdot \frac{u''\sqrt{(1-u''^2)}}{\sqrt{(1-\rho''\,u''^2}},$$

$$\rho^{\text{iv}} = \&\text{c.} \quad , \quad u^{\text{iv}} = \&\text{c.}$$

we shall have

$$\frac{d\,x}{\sqrt{(1-x^2)\,(1-\rho^2 x^2)}} =$$

$$\frac{1+\rho'}{2} \cdot \frac{1+\rho''}{2} \cdot \frac{1+\rho'''}{2} \cdots\cdots \frac{1+\text{P}}{2} \cdot \int \frac{d\,v}{\sqrt{(1-V^2)\,(1-\text{P}^2 V^2)}},$$

P and V designating respectively, the nth terms of the series ρ', ρ'', ρ''', &c. u', u'', u''', &c.

The advantage of this last form will be obvious, if we consider, that, since $\rho' = \dfrac{1 - \sqrt{(1 - \rho^2)}}{1 + \sqrt{(1 - \rho^2)}} = \dfrac{\rho^2}{[1 + \sqrt{(1 - \rho^2)}]^2}$,

ρ' must be less than the square of ρ, less therefore than the square of a fraction, for such ρ is always supposed to be: similarly, ρ'' must be less than the square of ρ'^2, ρ''', less than ρ''^2; and so on: the series, therefore, ρ', ρ'', ρ''', &c. must be a rapidly decreasing one; so that, a term P will be soon arrived at, so small as to enable us to neglect $P^2 V^2$.

Now if we may neglect $P^2 V^2$, the difficulty of finding the integral of $\dfrac{dx}{\sqrt{[(1 - x^2)(1 - \rho^2 x^2)]}}$ will be reduced to that of finding the integral of $\dfrac{dV}{\sqrt{(1 - V^2)}}$: for, the quantities ρ', ρ'', ρ''', &c. are easily computed.

The integral of $\dfrac{dV}{\sqrt{(1 - V^2)}}$, however, between the values of $V = 0$, and $V = 1$, is equal to $\dfrac{\pi}{2}$ $(\pi = 3.14159)$: the only question, therefore, that remains to be decided, is concerning the values of x, corresponding to the values of $V = 0$, and $V = 1$: and, in order to determine it, we must examine the values of u', u'', &c.

Now, $u' = \dfrac{2}{1 + \rho'} \cdot \dfrac{x \cdot \sqrt{(1 - x^2)}}{\sqrt{(1 - \rho^2 x^2)}}$;

consequently, $u' = 0$, both when $x = 0$, and when $x = 1$, and it is at its intermediate and maximum value, when

$$x = \frac{1}{\sqrt{[1 + \sqrt{(1 - \rho^2)}]}},$$

and, accordingly, the value of

$$\int \frac{du'}{\sqrt{[(1 - u'^2)(1 - \rho'^2 u'^2)]}},$$

between the values of $x = 0$, and $x = 1$, is *twice* the value of the same integral between $u' = 0$, and $u' = 1$: in like manner,

$$\int \frac{d u''}{\sqrt{[(1 - u''^2)(1 - \rho''^2 u''^2)]}},$$

contained between $u' = 0$, and $u' = 1$, is twice the value of the same integral contained between $u'' = 0$, and $u'' = 1$; and consequently, the value between $x = 0$, and $x = 1$, is four times the value of the same integral contained between $u'' = 0$, and $u'' = 1$: and so on for succeeding integrals. The value, then, of

$$\int \frac{d V}{\sqrt{(1 - V^2)}},$$

contained between the values of $x = 0$, and $x = 1$, is (2^n)th of the value $\left(\frac{\pi}{2}\right)$ of the same integral contained between $V = 0$, and $V = 1$. Hence, (see p. 287.)

$$\frac{a'}{2} A \pi = \frac{2}{1 + \rho'} \int \frac{dx}{\sqrt{[(1 - x^2)(1 - \rho^2 x^2)]}}$$

$$= \frac{2}{1 + \rho'} \cdot \frac{1 + \rho'}{2} \int \frac{du'}{\sqrt{(1 - u'^2)(1 - \rho'^2 u'^2)}}$$

$$= \&c.$$

$$= \frac{2}{1 + \rho'} \cdot \frac{1 + \rho'}{2} \cdot \frac{1 + \rho''}{2} \cdot \frac{1 + \rho'''}{2} \ldots\ldots \frac{1 + P}{2} \cdot \pi \times 2^n$$

$$= [(1 + \rho'')(1 + \rho''')(1 + \rho^{\text{IV}}) \ldots\ldots 1 + P] \cdot \pi,$$

and, accordingly,

$$\frac{A a'}{2} = (1 + \rho'')(1 + \rho''')(1 + \rho^{\text{IV}}) \ldots\ldots (1 + P),$$

which, considering the nature of the investigation, is an expression of remarkable simplicity.

We must now determine B, the coefficient of the second term.

Multiply each side of the equation of p. 286. by $\cos.\, \omega . d\omega$: then

$$\frac{1}{a'} \cdot \frac{\cos.\, \omega\, d\omega}{\sqrt{(1 - 2\rho' \cos.\, \omega + \rho'^2)}} =$$

$$\frac{1}{2}A \,.\, \cos.\,\omega \,.\, d\omega + B \,.\, \cos.^2\,\omega \,.\, d\omega + E \,.\, \cos.\,3\,\omega \,.\, \cos.\,\omega \,.\, d\,\omega + \text{\&c.}$$

But (see *Trig.* p. 26.),

$$\cos.\,n\omega \,.\, \cos.\,\omega = \frac{1}{2}[\cos.\,(n-1)\omega + \cos.(n+1)\omega];$$

$$\therefore \int \cos.\,n\omega \,.\, \cos.\,\omega \,.\, d\,\omega = \frac{\sin.\,(n-1)\,\omega}{n-1} + \frac{\sin.\,(n+1)\,\omega}{n+1},$$

and, consequently, the integral on the left-hand side of the preceding equation between the values of $\omega = 0$, and $\omega = \pi$, will equal 0 in every case, except in that of $n = 1$, and in that case, since $\cos.\,(1-1)\,\omega = \cos.\,0\,\omega = 1$, the above integral would be expressed by

$$\frac{1}{2}\int 1 \,.\, d\,\omega = \frac{\omega}{2} = \frac{\pi}{2} \text{ (}\omega\text{ becoming } = \pi\text{)}.$$

Every term then, except the one excepted, in the right-hand side of the equation of l. 1. becoming nothing after the integral has been supposed to be taken between $\omega = 0$, and $\omega = \pi$, there results

$$B\,a' . \frac{\pi}{2} = \int \frac{\cos.\,\omega \,.\, d\,\omega}{\sqrt{[(1 - 2\,\rho' \cos.\,\omega + \rho'^2)]}} \text{ between } \omega = 0, \text{ and } \omega = \pi,$$

and now, as before, the difficulty is reduced to that of finding an integral.

If we make the former substitution, namely, that of $x = \cos.\,\frac{\omega}{2}$, we shall have

$$B\,a'\,\frac{\pi}{2} = \frac{2}{1+\rho'}\int \frac{(2\,x^2 - 1)\,d\,x}{\sqrt{(1 - x^2)\,(1 - \rho^2\,x^2)}},$$

and from

$$u' = \frac{2}{1+\rho'} \,.\, x\sqrt{\left(\frac{1 - x^2}{1 - \rho^2 x^2}\right)},$$

we may deduce

$$2\,x^2 - 1 = \frac{\rho'}{2}\,(2\,\rho'^2 - 1) + \frac{\rho'}{2} - \sqrt{[(1 - u'^2)\,(1 - \rho'^2\,u'^2)]};$$

therefore, since (see p. 287.)

$$\frac{dx}{\surd[(1-x^2)(1-\rho^2x^2)]}=\frac{1+\rho'}{2}\cdot\frac{du'}{\surd[(1-u'^2)(1-\rho'^2u'^2)]},$$

$$\frac{(2x^2-1).dx}{\surd[(1-x^2)(1-\rho^2x^2)}=\frac{(1+\rho')\rho'}{2.2}\cdot\frac{(2u'^2-1).du'}{\surd[(1-u'^2)(1-\rho'^2u'^2)]}$$

$$+\frac{1+\rho'}{2}\cdot\frac{\rho'}{2}\frac{du'}{\surd[(1-u'^2)(1-\rho'^2u'^2)}-\frac{1+\rho'}{2}du'.$$

Now the first term on the right-hand side of the equation, involves a differential precisely similar to the differential on the left-hand side, and from which it was by transformation derived. The process, therefore, as in p. 289. may be repeated, and like results, or similar differentials, &c. obtained, by means of those values of u'', u''', and ρ'', ρ''', &c. which are given in p. 287. accordingly,

$$\frac{(2u'^2-1)du'}{\surd(1-u'^2)(1-\rho'^2u'^2)}=\frac{1+\rho''}{2}\cdot\frac{\rho''}{2}\,\frac{(2u''^2-1).du''}{\surd[(1-u''^2)(1-\rho'''u''^2)}$$

$$+\frac{1+\rho''}{2}\cdot\frac{\rho''}{2}\,\frac{du''}{\surd[(1-u''^2)(1-\rho''^2u''^2)}$$

$$-\frac{1+\rho''}{2}.du''$$

similarly,

$$\frac{(2u''^2-1)\,du''}{\surd[(1-u''^2)(1-\rho'^2u''^2)}=\&c.$$

so that

$$\frac{(2x^2-1)dx}{\surd[(1-x^2)(1-\rho^2x^2)]}=$$

$$\frac{(1+\rho')(1+\rho'')(1+\rho''')..(1+\mathrm{P})}{2.2.2......2}\cdot\frac{\rho'\rho''\rho'''....\mathrm{P}}{2\,2.2....2}\cdot\frac{dV}{\surd(1-V^2)}$$

$$+\left\{\begin{array}{l}\dfrac{\rho'.(1+\rho')(1+\rho'')....(1+\mathrm{P})}{2.2.2........2}\\[2ex] +\dfrac{\rho'\rho''.(1+\rho')(1+\rho'').....(1+\mathrm{P})}{2.2.2.2......2}\\[2ex] +\dfrac{\rho'\rho''\rho'''.(1+\rho')..\;...(1+\mathrm{P})}{2.2.2.2........2}\end{array}\right\}\times\frac{dV}{\surd(1-V^2)}$$

$$-\left(\frac{1+\rho'}{2}.du'+\frac{(1+\rho')(1+\rho'')\rho'}{2.2.2}du''+\&c.\right).$$

Now, it is to be observed, of the preceding terms the first term will disappear, (or may, from its minuteness, be neglected) by reason of the factor,

$$\frac{\rho' \cdot \rho'' \cdot \rho''' \ldots\ldots \mathrm{P}}{2\,.\,2\,.\,2 \ldots\ldots 2}\,.$$

which becomes very minute: and, besides, the last term will disappear, when the whole integral is taken between the values of $x = 0$, and $x = 1$, because all the quantities u', u'', u''', &c. will then become nothing. The middle term then solely remains after the whole integral has been taken; and, since

$$\int \frac{dV}{\surd(1 - V^2)} = 2^n \cdot \frac{\pi}{2};$$

there results

$$\frac{Ba'\pi}{2} = \frac{1}{1+\rho'}\left(\frac{\rho'}{2} + \frac{\rho' \cdot \rho''}{2\,.\,2} + \frac{\rho' \cdot \rho'' \cdot \rho'''}{2\,.\,2\,.\,2} + \&\text{c.}\right)(1+\rho')(1+\rho'')\,.\,.(1+\mathrm{P})\pi,$$

or,

$$Ba' = (1+\rho'')\,(1+\rho''')\ldots.(1+\mathrm{P})\left(\rho' + \frac{\rho' \cdot \rho''}{2} + \frac{\rho' \cdot \rho'' \cdot \rho'}{2\,.\,2} + \&\text{c.}\right),$$

an expression as remarkable as the preceding one, (see p. 289.) for A, and admitting, in specific cases, of an equally easy numerical computation.

We have now gone through that part of the investigation by which the two first coefficients A and B are found. The remaining part, namely, the derivation of C, D, E, &c. from the two first A and B, is of a less intricate nature. We will now enter on it.

Let the coefficients of cos. $(m-2)\,\omega$, cos. $(m-1)\,\omega$, cos. $m\omega$, be K, L and M respectively, then, according to the methods of p. 290.

$$Ka' \cdot \frac{\pi}{2} = \int \frac{\text{cos.}\,(m-2)\,\omega \cdot d\omega}{\surd(1 - 2\rho' \cdot \text{cos.}\,\omega + \rho'^2)},$$

$$La' \cdot \frac{\pi}{2} = \int \frac{\text{cos.}\,(m-1)\,\omega \cdot d\omega}{\surd(1 - 2\rho' \cdot \text{cos.}\,\omega + \rho'^2)},$$

$$Ma' \cdot \frac{\pi}{2} = \int \frac{\text{cos.}\,m\,\omega \cdot d\omega}{\surd(1 - 2\rho' \cdot \text{cos.}\,\omega + \rho'^2)},$$

the three integrals being supposed to be taken between the values of $\omega = 0$, and $\omega = \pi$.

Now,

$$\int [\cos. (m-1)\,\omega\, \surd(1 - 2\,\rho'. \cos.\,\omega + \rho'^2)\, d\,\omega) =$$

$$\frac{\sin. (m-1)\,\omega\, \surd(1 - 2\,\rho' \cos.\,\omega + \rho'^2)}{m-1} -$$

$$\frac{\rho'}{m-1}.\int\left(\frac{\sin. (m-1)\,\omega\,.\,\sin.\,\omega}{\surd(1 - 2\rho' \sin.\,\omega + \rho'^2)}\,d\,\omega\right),$$

and, since

$$\sin. (m-1).\omega.\sin.\,\omega = \frac{1}{2}\,[\cos. (m-2)\,\omega - \cos.\,m\,\omega],$$

$$\rho'.\int\left(\frac{\sin. (m-1)\,\omega\,.\sin.\,\omega\,.\,d\,\omega}{\surd(1 - 2\,\rho'.\cos.\,\omega + \rho'^2)}\right) = \text{(see p. 292. ll. 26, 28.)}$$

$$\frac{\rho'}{2}\,K\,a'\frac{\pi}{2} - \frac{\rho'}{2}\,M\,a'.\frac{\pi}{2}.$$

Again,

$$\int [\cos. (m-1)\,\omega\,.\,\surd(1 - 2\rho' \cos.\,\omega + \rho'^2)\,d\,\omega] =$$

$$(1+\rho'^2)\int\frac{\cos. (m-1)\,\omega\,.\,d\,\omega}{\surd(1 - 2\rho'.\cos.\,\omega + \rho'^2)} -$$

$$\rho'.\int\left(\frac{\cos.\,m\,\omega + \cos. (m-2)\,\omega}{\surd(1 - 2\,\rho'.\cos.\,\omega + \rho'^2)}\,d\,\omega\right)$$

$$= (1+\rho'^2)\,L\,a'.\frac{\pi}{2} - M\,a'\rho'.\frac{\pi}{2} - K\,a'\rho'.\frac{\pi}{2}.$$

Now, $\sin. (m-1)\,\omega = 0$, when $\omega = 0$: dividing, therefore, each term by $a'.\frac{\pi}{2}$, we have from ll. 10, 15.

$$L.(1+\rho'^2) - M\,\rho' - K\,\rho' = \frac{M\,\rho'}{2.(m-1)} - \frac{K\,\rho'}{2.(m-1)};$$

whence,

$$M = -\frac{2\,m-3}{2\,m-1}\,K + \frac{2\,m-2}{2\,m-1}\,.\,\frac{1+\rho'^2}{\rho'}.L,$$

which, as it is evident, is a general formula for any coefficient in terms of the two preceding coefficients; and which, consequently, enables us to determine all the coefficients of the expanded form for

$$\frac{1}{\sqrt{(a'^2 - 2aa' \,.\, \cos.\, \omega + a^2)}},$$

from the two first A and B: for instance, if

$$m = 2,$$

$$C = -\frac{A}{2} + \frac{2}{3} \,.\, \frac{1+\rho'^2}{\rho'} \,.\, B,$$

$$m = 3,$$

$$D = -\frac{3}{5} B + \frac{4}{5} \,.\, \frac{1+\rho'^2}{\rho'} \,.\, C,$$

$$m = 4,$$

$$E = -\frac{5}{7} C + \frac{6}{7} \,.\, \frac{1+\rho'^2}{\rho'} \,.\, D,$$

&c.

These are the values of C, D, E, &c.; but, if we revert to the values of R and of $r\frac{dR}{dr}$, see pp. 279, &c. we shall find that it is necessary to know the values of $\frac{dA}{da}$, $\frac{dB}{da}$, &c. $\frac{d^2A}{da^2}$, &c. In order to determine their values, let us resume the equations of p. 286.

$$\frac{1}{a'\sqrt{(1-2\rho' \,.\, \cos.\, \omega + \rho'^2)}} = \frac{1}{2} A + B \,.\, \cos.\, \omega + C \,.\, \cos.\, 2\,\omega + \&c.$$

$$\therefore \left(\frac{1}{a'^2} \cos.\, \omega - \frac{a}{a'^3}\right) (1 - 2\rho' \,.\, \cos.\, \omega + \rho'^2)^{-\frac{3}{2}} =$$

$$\frac{1}{2} \,.\, \frac{dA}{da} + \frac{dB}{da} \cos.\, \omega + \frac{dC}{da} \cos.\, 2\,\omega + \&c.$$

from which equation it is plain that the differential coefficients $\frac{dA}{da}$, $\frac{dB}{da}$, &c. might be determined, if the expanded form of $(1 - 2\rho' \,.\, \cos.\, \omega + \rho'^2)^{-\frac{3}{2}}$ were known. Assume

$$\frac{1}{a'^3 y^3}\left(=\frac{1}{a'^3(1-2\rho'\cos.\omega+\rho'^2)^{\frac{3}{2}}}\right)=\frac{1}{2}A'+B'.\cos.\omega+C'.\cos.2\omega+\&c.$$

then, as before, (see pp. 290. 292.)

$$M'a'^3.\frac{\pi}{2}=\int\frac{\cos.m\omega}{y^3}d\omega,$$

the integral being taken between the values of $\omega=0$, and $\omega=\pi$.

Again, as in p. 293,

$$\int\frac{\cos.(m-1)\omega.d\omega}{y^3}=$$

$$\frac{\sin.(m-1)\omega}{(m-1).y}+\frac{\rho'}{m-1}\int\frac{\sin.(m-1)\omega.\sin.\omega}{y^3},$$

or, see p. 292, &c.

$$L=\frac{K'\rho'a'^2}{2.(m-1)}-\frac{M'\rho'a'^2}{2.(m-1)}\ldots\ldots\ldots\ldots[a].$$

But,

$$\frac{\cos.(m-1)\omega}{y}=\frac{\cos.(m-1)\omega.y^2}{y^3}$$

$$=\frac{1+\rho'^2}{y^3}-\frac{2\rho'.\cos.(m-1)\omega.\cos.\omega}{y^3};$$

consequently,

$$\int\frac{\cos.(m-1)\omega}{y}d\omega\,(=La')=(1+\rho'^2)L'a'^3.\frac{\pi}{2}-M'\rho'a'^3\frac{\pi}{2}$$

$$-K\rho'.a'^3.\frac{\pi}{2}.$$

Now, if we equate this second value of La' with the former of l. 10., we shall have

$$(1+\rho'^2)L'-M'\rho'-K'\rho'=\frac{K'\rho'}{2.(m-1)}-\frac{M'\rho'}{2.(m-1)},$$

and

$$M'=\frac{2m-2}{2m-3}.\frac{L'.(1+\rho'^2)}{\rho'}-\frac{2m-1}{2m-3}K'\ldots\ldots\ldots[a].$$

If we combine this equation with the equation [a], and respectively eliminate M' and K', there will result,

$$L = \frac{2\,K'\,\rho'.\,a'^2}{2m-3} - \frac{L'.\,(1+\rho'^2)\,a'^2}{2\,m-3},$$

$$L = \frac{L'\,(1+\rho'^2)\,a'^2}{2\,m-1} - \frac{2\,M'\,\rho'\,a'^2}{2\,m-1},$$

$$\text{or, } L = \frac{2\,K'\,a\,a'}{2\,m-3} - \frac{L'.\,(a^2+a'^2)}{2\,m-3},$$

$$\text{and } L = \frac{L'.\,(a^2+a'^2)}{2m-1} - \frac{2\,M'\,a\,a'}{2m-1} \ldots\ldots\ldots\ldots\ldots [b].$$

If in the equation (b) we write m instead of $m-1$, and M, K', L', instead of L, K', and L', we shall have

$$M = \frac{2\,L'.\,a\,a'}{2m-1} - \frac{M'.\,(a^2+a'^2)}{2\,m-1} \ldots\ldots\ldots\ldots\ldots [c].$$

and lastly, if we determine L' and M' from the two equations (b) and (c), there will result,

$$L' = (2\,m-1)\left(\frac{L.\,(a^2+a'^2)}{(a^2-a'^2)^2} - \frac{2\,M\,a\,a'}{(a^2-a'^2)^2}\right) \ldots\ldots\ldots [e].$$

$$M' = (2m-1)\left(\frac{2\,L\,a\,a}{(a^2-a'^2)^2} - \frac{M\,(a^2+a'^2)}{(a^2-a'^2)^2}\right) \ldots\ldots [f].$$

and thus the coefficients of the terms of the development of $\frac{1}{(a'^2-2\,a\,a'.\cos.\,\omega+a^2)^{\frac{3}{2}}}$ are expressed in terms of the development of $\frac{1}{\sqrt{(a'^2-2\,a\,a'\cos.\,\omega+a^2)}}$.

If in the second of the equations (b), $m=1$, L will be A, and L', M', will be A' and B' respectively, and then

$$A = A'.\,(a^2+a'^2) - 2\,B'.\,a\,a',$$

make $m=2$, and from the first of the equations (b), we shall have

$$B = 2\,A'\,a\,a' - B'\,(a^2+a'^2),$$

which values agree with those which Lagrange has given in his

Mec. Anal. edit. 2. Seconde Partie. Sect. VII. p. 141. and, in the same manner, we may deduce from the equations (*e*) and (*f*),

$$A' = \frac{A.(a^2 + a'^2)}{(a^2 - a'^2)^2} - \frac{2\,B\,a\,a'}{(a^2 - a'^2)^2},$$

$$B' = \frac{2\,A\,a\,a'}{(a^2 - a'^2)^2} - \frac{B\,.(a^2 + a'^2)}{(a^2 - a'^2)^2},$$

which agree with those given by Laplace in the *Mec. Cel.* Premiere Partie, Liv. II. p. 269.

We are now possessed, then, (see p. 294.) of a simple method of determining the partial differential coefficients $\frac{dA}{da}$, $\frac{dB}{da}$, &c. : for by the equation of p. 294. l. 19.)

$$(a'.\cos.\omega - a)\left\{\begin{matrix}\frac{A'}{2} + B'\cos.\omega + C'.\cos.2\omega + \&c. \\ + L'.\cos.(m-1)\,\omega + \&c.\end{matrix}\right\} =$$

$$\frac{1}{2}.\frac{dA}{da} + \frac{dB}{da}\cos.\omega + \frac{dC}{da}\cos.2\omega + \&c. \frac{dL}{da}\cos.(m-1)\omega + \&c.$$

and hence, by equating the coefficients of the cosines of like arcs,

$$\frac{dB}{da} = \frac{C'a'}{2} - B'a + \frac{A'a'}{2},$$

$$\frac{dC}{da} = \frac{D'a'}{2} - C'a + \frac{B'a'}{2},$$

$$\&c. = \&c.$$

and generally,

$$\frac{dL}{da} = \frac{M'a'}{2} - L'a + \frac{K'a'}{2}$$

In this equation substitute for K' its value as contained in the equation (α) of p. 295, and

$$\frac{dL}{da} = \frac{1}{2m-1} \cdot \left(\frac{(m-1) . a'^2 - ma^2}{a} L' + M' a' \right).$$

Lastly, substitute instead of L' and M', those their values which are contained in the equations (e), (f) of p. 296. and there will result,

$$\frac{dL}{da} = \frac{(m-1)\, a'^2 + m\, a^2}{a \, . (a'^2 - a^2)} L - \frac{(2m-1) \, . \, M a'}{a'^2 - a^2},$$

which is a general expression.

In instances, if, $m = 1$,

$$\frac{dA}{da} = \frac{A a}{a'^2 - a^2} - \frac{B a'}{a'^2 - a^2},$$

$$m = 2, \qquad \frac{dB}{da} = \frac{a'^2 + 2a^2}{a . (a'^2 - a^2)} B - \frac{3 C a'}{a'^2 - a^2},$$

and if we substitute for C its value in terms of A and B (see p. 294.)

$$\frac{dB}{da} = \frac{A a a' - B a'^2}{a \, . \, (a'^2 - a^2)}.$$

In a like manner $\frac{dC}{da}$, $\frac{dD}{da}$, &c may be computed.

We possess then, whatever the value of $\frac{a}{a'}$ be, the means of computing A, B, &c. $\frac{dA}{da}$, $\frac{dB}{da}$, &c. on which (see p. 279.) the value of R depends; and, that value being determined, the perturbation in parallax and longitude, may, as it has been already shewn, (pp. 258, 268.) be determined.

In the Lunar Theory the formulæ of the present Chapter are of no use, or rather, they are not required. They are also not required in several other cases of planetary disturbance: but they are applicable to all cases, and, once being invented, they ought, since compendium of calculation is a thing much to be desired, to be used. There is no good reason for restricting them to their special uses.

The mutual perturbations of Jupiter and Saturn, of Venus and the Earth require, as has been already stated, these special uses. But they would be most useful, and, indeed, indispensably necessary, in a research of the mutual perturbations of Ceres and Pallas *.

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* In this case, since $\frac{a}{a'} = 1$ nearly, the ordinary methods of computing *A* and *B* (see pp. 260, &c.) would be altogether useless: the formulæ, however, of pp. 289, 292. would even then apply, since, if $\frac{a}{a'}$ be at all less than 1, the terms ρ', ρ'', ρ''', &c. must decrease: but they would $\left(\frac{a}{a'}\text{ being nearly equal } 1\right)$ decrease very slowly. For the purpose of procuring greater expedition of computation, the Author of the present Treatise deduced, in the *Phil. Trans.* for the year 1804, (p. 219.) a method for computing the two first coefficients *A* and *B*, (the index being either $\frac{1}{2}$, or $\frac{3}{2}$), which is most useful when a is most nearly equal to a'. In fact, the principal series, on which the computation depends, involves, as it is produced, more and more products, such as $b'b''$, $b'b''b'''$, &c. b being equal $\sqrt{(1-\rho'^2)} = \sqrt{\left[1-\left(\frac{a}{a'}\right)^2\right]}$, and the law for b', b'', &c. being

$$b'' = \frac{1-\sqrt{(1-b'^2)}}{1+\sqrt{(1-b'^2)}},$$

$$b''' = \frac{1-\sqrt{(1-b''^2)}}{1+\sqrt{(1-b''^2)}},$$

&c.

See *Phil. Trans.* 1804, pp. 246, &c.

The subject of the present Chapter has been fully and frequently treated of by foreign mathematicians. The investigation began with Euler in his *Recherches sur Jupiter et Saturne,* and has been continued to the

What is peculiar in the present Chapter, is the method of computing the coefficients of the two first terms of the development of $\frac{1}{(a'^2 - 2aa' \cos. \omega + a^2)^{\frac{1}{2}}}$: that alone, therefore, requires an illustration: and we will endeavour to find one in the perturbation of the Solar Orbit by the action of Venus.

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the present time. Analytical Science, as it is called, has been much benefited by these investigations; and, as in numerous other instances, the instrument of calculation has been improved, because an improvement in it was wanted.

The mere wants and demands, however, of Physical Science, although they should always direct, are not to restrict the progress of analytical calculation. There is no defining the limits between useful formulæ and formulæ merely curious. During the advancement of science, the latter are continually changing their nature, and move from their class into that of the useful formulæ. The present researches are a proof of it. The doctrine of integrals or fluents, and the properties of quantities such as $\frac{dx}{\sqrt{(1-x^2)(1-e^2x^2)}}$, $dx\sqrt{\left(\frac{1-e^2x^2}{1-x^2}\right)}$ deduced not for an occasion like the present, but for establishing certain curious relations between elliptic arcs, have enabled us to compute the action of Venus on the Earth, and to improve the Solar Tables. There is much food for speculation in these matters; and (see pp. 286, &c.) a specimen has already been afforded of the manner by which the refined, and seemingly abstruse, methods of Analytics become subservient to Physical Science.

See on the subject of this Chapter. Euler, *Recherches sur les Inegalités de Jupiter et Saturne:* Clairaut, *Mem. Acad. des Sciences*, 1754, pp. 546, &c. Dalembert, *Recherches sur differens Points importans du Systéme du Monde*, pp. 660, &c. Lagrange, *Berlin Acts*, 1781, pp. 252, &c. and *Mec. Analyt.* ed. 2. pp. 141, &c. Laplace, *Mec. Cel.* Premiere Part. Liv. II. pp. 267, &c. Legendre, *Mem. de l'Acad.* 1786, pp. 663, &c. Lacroix, *Calc. Diff.* &c. vol. II. ed. 1. p. 454. Ivory, *Edin. Trans.* vol. IV. pp. 178. Wallace, *Edin. Trans.* vol. V. pp. 280, &c. Woodhouse, *Phil. Trans.* 1804. pp. 219, &c.

Computation of the Coefficients A *and* B.

In order to expedite the computation, assume $\rho' = \sin.\theta'$, $\rho'' = \sin.\theta''$, $\rho''' = \sin.\theta'''$, &c.

then, (see p. 287.) $\rho'' = \dfrac{1 - \sqrt{(1-\rho'^2)}}{1 + \sqrt{(1-\rho'^2)}} =$

$\dfrac{1 - \cos.\theta'}{1 + \cos.\theta'} =$ (see *Trig.* p. 28.) $\tan.^2 \dfrac{\theta'}{2}$,

similarly,

$$\rho''' = \tan.^2 \frac{\theta''}{2}, \quad \rho^{\text{iv}} = \tan^2. \frac{\theta'''}{2}, \ \&c.$$

whence,

$$1 + \rho'' = \sec.^2 \frac{\theta'}{2},$$

$$1 + \rho''' = \sec.^2 \frac{\theta''}{2},$$

$$1 + \rho^{\text{iv}} = \sec.^2 \frac{\theta'''}{2},$$

$$\&c. = \&c.$$

and (see p. 289.)

$$\log. \frac{A a'}{2} = 2\left(\log.\sec.\frac{\theta'}{2} + \log.\sec.\frac{\theta''}{2} + \&c.\right) - (20 + 20 + \&c.)$$

$\frac{1}{2}$ *A computed.*

Now, $\rho' = \dfrac{a}{a'}$ .... = .7233323 :

the logarithm of which is ....9.853379, and the arc (θ'), the logarithmic sine of which is 9.853379, is 46° 19′ 49″.

Hence, $\frac{\theta'}{2} = 23^{\circ}\ 9'\ 54''.5$,

$$\log.\tan.\frac{\theta'}{2}\ldots\ldots\ldots\ldots\ldots\ldots 9.6313225\ldots\log.\sec.\frac{\theta'}{2}\ 10.0365073$$

$$\therefore \log.\sin.\theta''\left(=2\log.\tan.\frac{\theta'}{2}\right)9.2626450$$

and $\theta'' = 10^{\circ}\ 32'\ 57''.5$.

Again,

$$\log.\tan.\frac{\theta''}{2}\ldots\ldots\ldots\ldots\ldots\ldots\ 8.9653005\ldots\log.\sec.\frac{\theta''}{2}\ 10.0018429$$

$$\therefore \log.\sin.\theta'''\left(=2\log.\tan.\frac{\theta''}{2}\right)7.9306010$$

and $\theta''' = 0^{\circ}\ 29'\ 18''$.

Again,

$$\log\tan.\frac{\theta'''}{2}\ldots\ldots\ldots\ldots\ldots\ldots\ 7.6295664\ldots\log.\sec.\frac{\theta'''}{2}\,10.0000039$$

$$\text{(sum of the log. secants) } 30.0383541$$

$$\therefore \log.\frac{A\,a'}{2} = 60.0767082 - (20 + 20 + 20)$$

$$= .0767082,$$

$$\text{and } \frac{A\,a'}{2} = 1.193189.$$

It appears from the process itself, that, from their rapid decrease, the computation of three logarithmic secants is sufficiently exact.

B *computed.*

$$B\,a' \text{ (see p 292.)} = (1+\rho'')(1+\rho''')\left(\rho' + \frac{\rho'\rho''}{2} + \frac{\rho'\rho''\rho'''}{2\,.\,2} + \&c.\right)$$

$$= \frac{A\,a'}{2}\,.\left(\rho' + \frac{\rho'\rho''}{2} + \frac{\rho'\rho''\rho'''}{2\,.\,2} + \&c.\right)$$

Now,

$\log.\ \rho' = 9.8593379 \ldots\ldots\ldots\ldots \rho' = .723332,$

$\log.\ \rho'' = \underline{9.2626450}$

$(\log.\ \rho'\rho'') = 9.1219829 \ldots\ldots\ldots \frac{1}{2}\rho'\rho'' = .066214.$

Again,

$\log.\ \rho'''\ \ \underline{7.9306010}$

$(\log.\ \rho'\rho''\rho''')\ 7.0525839 \ldots\ldots\ldots \frac{1}{4}\rho'\rho''\rho''' = \underline{.000282}$

$.789828$

$\log.\ .789828 \ldots\ldots\ldots 9.8975865$

$\log.\ \frac{Aa'}{2} \ldots\ldots\ldots\ldots \underline{0.0767082}$

$9.9742947\ \ (= \log.\ Ba')$

$\therefore\ Ba' = .94252.$

Laplace's numbers for the coefficients corresponding to $\frac{Aa'}{2}$, and Ba' are 1.193172, and .942413, respectively.

The other coefficients C, D, E, &c. may be deduced (see pp. 294, &c.) from A and B, by the most simple of arithmetical operations.

In order not to interrupt the course of deduction which led from the most simple to the most complex case of the formation of the coefficients A, B, C, &c. we did not complete, as we had proposed to do, (see p. 272.), the computation of the Earth's perturbations from the action of Jupiter. We will now resume that subject: and, instead of those values of $\frac{1}{2}A$ and B which were given in p. 283, we will take

$$\frac{1}{2}Aa' = 1.009442, \quad Ba' = .195003,$$

two values which may be derived from that method which has just been described and exemplified.

From these two first coefficients, the others, as it has been shewn in p. 294, may be computed, and C and D from these expressions,

$$C = -\frac{A}{3} + \frac{2}{3} \cdot \frac{1+\rho'^2}{\rho'} B,$$

$$D = -\frac{3}{5} B + \frac{4}{5} \cdot \frac{1+\rho'^2}{\rho'^2} C;$$

whence,

$$C a' = .02819, \quad D a' = .0046\,^*.$$

---

* Thus computed:

$\rho' = .1222646$ .... log. $\rho' = 9.2838993$

$\rho'^2 = .0369645$.........

$1 + \rho'^2 = 1.0369656$... log. $\frac{2}{3}(1+\rho'^2) = 9.8396728$

log. $B a' = 9.2900413$

19.1297141

log. $\rho' = 9.2838993$

$9.8458148 = \text{log.}\ \frac{2}{3} B a' . \frac{1+\rho'^2}{\rho'}$;

$\therefore \frac{2}{3} B a' \frac{1+\rho'^2}{\rho'} = .70115$

and $\frac{1}{3} A a' = .67296$

$\therefore C a' = .02819$

Again,

log. $C a' = 8.4500951$

log. $\frac{4}{5}(1+\rho'^2) = 9.9188543$

18.3689494

log. $\rho' = 9.2838993$

$9.0850501 = \text{log.}\ \frac{4}{5} C a' . \left(\frac{1+\rho'^2}{\rho'}\right)$;

$\therefore \frac{4}{5}$

In order to find $\frac{dB}{da}$, $\frac{dC}{da}$, we have, see p. 298.

$$\frac{dB}{da} = \frac{Aa'}{a'^2 - a^2} - \frac{Ba'^2}{a \,.\, (a'^2 - a^2)},$$

$$\text{or, } a'^2 . \frac{dB}{da} = \frac{\mathrm{A}a'}{1 - \rho'^2} - \frac{\mathrm{B}a'}{\rho' \,.\, (1 - \rho'^2)};$$

$$\text{whence, } \quad a'^2 . \frac{dB}{da} = 1.0432,$$

* and by a similar process,

$$\therefore \ \tfrac{4}{5}\, Ca' . \frac{1 + \rho'^2}{\rho'} = .12163,$$

$$\tfrac{3}{5}\, Ba' \quad \underline{.11700}$$

$$\therefore Da' = .00463$$

* Computation.

$$\log. Ba' = 9.2900413$$

$$\log. \frac{1}{\rho'} = \underline{.7161006}$$

$$10.0061419$$

$$\log. (1 - \rho'^2) = \underline{.9836418}$$

$$.0225001$$

$$(n) \,.\, \frac{Ba'}{\rho' (1 - \rho'^2)} = 1.0532.$$

$$\log. Aa' = .3051098$$

$$\log. (1 - \rho'^2) = \underline{9.9836418}$$

$$.321468 = \log. \frac{Aa'}{1 - \rho'^2};$$

$$\therefore \frac{Aa'}{1 - \rho'^2} = 2.0964$$

$$(n) = \underline{1.0532}$$

$$\therefore a'^2 . \frac{dB}{da} = 1.0432$$

$$a'^2 . \frac{d\,C}{d\,a} = .297995.$$

The numerical values of these coefficients are sufficient for the computation of the coefficients of the two principal terms of the variation: that is, in the present case, of the terms the arguments of which are $n't - nt + \epsilon' - \epsilon$, and $2\,(n't - nt + \epsilon' - \epsilon)$.

Let ω represent the angle $n't - nt + \epsilon' - \epsilon$, and assume, as in p. 274.

$$R = m' . \left(\frac{1}{2}\,\text{A} + \text{B} \cos.\,\omega + \Gamma \cos.\,2\,\omega + \&c.\right)$$

Hence, as in p. 264,

$$r\,.\,\frac{d\,R}{d\,r} = a\,.\,\frac{d\,R}{d\,a} =$$

$$m'\left(\frac{a}{2}\,.\,\frac{d\,\text{A}}{d\,a} + a\,.\,\frac{d\,\text{B}}{d\,a}\cos.\,\omega + a\,.\,\frac{d\,\Gamma}{d\,a}\cos.\,2\,\omega + \&c.\right),$$

and $$2\int\frac{d\,R}{d\,r} = \frac{2\,m'\,n}{n - n'}\,.\,(\text{B}\cos.\,\omega + \Gamma\,.\,\cos.\,2\,\omega + \&c.)$$

and accordingly,

$$\frac{r\,d\,R}{d\,r} + 2\int d\,R =$$

$$\frac{m'}{2}\,.\,a\,.\,\frac{d\,\text{A}}{d\,a} + m'\,.\left(a\frac{d\,\text{B}}{d\,a} + \frac{2\,n\,\text{B}}{n - n'}\right)\cos.\,\omega$$

$$+ m'\left(a\,.\,\frac{d\,\Gamma}{d\,a} + \frac{2\,n\,\Gamma}{n - n'}\right)\cos.\,2\,\omega$$

$$+ \&c.$$

If we substitute this value in the equation of p. 258, and divide the equation by $a^2\left(= \frac{a^3}{a} = \frac{1}{n^2\,a}\right)$, there will result,

$$0 = \frac{d^2.r\delta r}{a^2.dt^2} + N^2\,.\,\frac{r\,\delta\,r}{a^2} + \&c.$$

$$+ n^2\,m'\left(a^2\,.\,\frac{d\,\text{B}}{d\,a} + \frac{2\,n}{n - n'}\,a\,\text{B}\right)\cos.\,\omega$$

$$+ n^2\,m'\left(a^2\,.\,\frac{d\,\Gamma}{d\,a} + \frac{2\,n}{n - n'}\,.\,a\,\Gamma\right)\cos.\,2\,\omega$$

$$+ \&c.$$

If we integrate this equation, neglecting the terms that are not periodical (see pp. 264. 269.),

$$\frac{r\,\delta r}{a^2} = m'\left\{\begin{array}{l} \dfrac{n^2}{(n-n')^2 - N^2}\left(a^2 \cdot \dfrac{d\mathrm{B}}{da} + \dfrac{2n}{n-n'} \cdot a\mathrm{B}\right)\cos.\,\omega \\ + \dfrac{n^2}{4(n-n')^2 - N^2}\left(a^2 \cdot \dfrac{d\Gamma}{da} + \dfrac{2n}{n-n'}\,a\Gamma\right)\cos.\,2\omega \\ + \;\&\text{c.} \end{array}\right\}$$

which value is now deduced as being subservient to the deduction of δv (see pp. 268, 269).

The terms which depend on the angular distance of the Earth and Jupiter, or, which have for their arguments, ♃ − ⊖, 2 (♃ − ⊖), &c. (arguments analogous to those that form the Lunar *Variation*, see pp. 217, &c.) are independent of the eccentricity: but dr involves e; therefore, as in p. 269. the term $\dfrac{dr \,.\, \delta r}{a^2 \,.\, n\,dt}$ in the value of δv may be set aside, and we shall have, for the computation of that inequality in longitude which is independent of the eccentricity, this formula (making $\mu = 1$),

$$\delta v = \frac{2 \,.\, d\,(r\,\delta r)}{a^2 \,.\, n\,dt} + 3a\int\!\!\int n\,dt \,.\, dR + 2a\int n\,dt \,.\, r \,.\, \frac{dR}{dr}\,.$$

Let the term in R of which the argument is $p\omega$ be $m'P\cos.\,p\omega$, then (see pp. 97, 265.) the term in $\dfrac{r\,\delta r}{a^2}$, which has the same argument, is

$$\frac{m'\,n^2}{p^2 \,.\, (n-n')^2 - N^2} \cdot \left(a^2 \,.\, \frac{dP}{da} + \frac{2n}{n-n'} \,.\, aP\right)\cos.\,p\omega\,;$$

find, with regard to those terms, the value of δv from the above expression, precisely as it was found in p. 269., and there will result

$$\delta v =$$

$$\frac{m'n}{n-n'}\left[\frac{1}{p}\Big(\frac{n}{n-n'}aP + \frac{2N^2}{p^2(n-n')^2 - N^2}\Big(a^2 \,.\, \frac{dP}{da} + \frac{2n}{n-n'} \,.\, aP\Big)\right]\cos.\,p\omega,$$

or, in its more expanded form, (making $p = 1,\ 2$, &c.)

$$\delta v =$$

$$\frac{m'n}{n-n'} \cdot \left\{ \begin{array}{l} \left[\frac{n}{n-n'} \cdot a\mathrm{B} + \frac{2N^2}{(n-n')^2 - N^2}\left(a^2 \cdot \frac{d\mathrm{B}}{da} + \frac{2n}{n-n'} \cdot a\mathrm{B}\right)\right] \cos. \omega \\ + \frac{1}{2}\left[\frac{n}{n-n'} a\Gamma + \frac{2N^2}{4(n-n')^2 - N^2}\left(a^2 . \frac{d\Gamma}{da} + \frac{2n}{n-n'} . a\Gamma\right)\right] \cos. 2\omega \\ + \&c. \end{array} \right\}$$

from which expression the constant parts, that would be introduced by the process of integration, are, for the reason stated in p. 269., omitted.

What now remains to be done is to find the numerical values of the preceding terms.

Since

$$R = m'\left(\frac{1}{2}\mathrm{A} + \mathrm{B} \cos. \omega + \Gamma \cos. 2\omega + \&c.\right)$$

$$= \text{(see p. 279.)}$$

$$\frac{m'a}{a'^2} \cos. \omega - m'\left(\frac{1}{2}A + B \cos. \omega + C . 2\omega + \&c.\right)$$

$$\mathrm{A} = -A, \quad \mathrm{B} = \frac{a}{a'^2} - B, \quad \Gamma = -C,$$

$$\frac{d\mathrm{B}}{da} = \frac{1}{a'^2} - \frac{dB}{da}, \&c.$$

Hence, since $\frac{a}{a'} = \rho' = .1922646$, $\frac{n}{n'} = \frac{12959577.35}{109256''.29}$

and m' (♃'s mass) $= \frac{1}{1067.09}$, we have

$$\frac{n}{n-n'} a\mathrm{B} = \frac{n}{n-n'} . \rho'^2 - \rho' \times .195003 = -.000575$$

therefore, twice that value............ $= -.00115$

Again,

$$a^2 . \frac{d\mathrm{B}}{da} = (\rho'^2 - \rho'^2 \times 1.0432) \ldots\ldots = -.001597$$

$$\therefore a^2 . \frac{d\mathrm{B}}{da} + \frac{2n}{n-n'} a\mathrm{B} \ldots\ldots\ldots\ldots\ldots = -.002747$$

Again, since $\frac{2\,N^2}{(n-n')^2 - N^2} = \frac{2\,n^2}{-\,n' \, . \, (2\,n - n')}$, (nearly),

$$= -\,2\,\frac{n^2}{n'^2}\left(\frac{1}{\frac{2\,n}{n'} - 1}\right),$$

$$\frac{2\,N^2}{(n-n')^2 - N^2}\left(a^2 \, . \, \frac{d\mathrm{B}}{d\,a} + \frac{2\,n}{n - n'}\,a\,\mathrm{B}\right) = .034023$$

$$\text{but } \frac{n}{n-n'} \, . \, a\,\mathrm{B} \ldots\ldots\ldots = -\ .000575$$

$\therefore$ (S) sum of two last lines .... = .033448

and lastly, (see p. 308. ll. 2, 3.)

$$* \frac{m'\,n}{n - n'} \, . \, S = 0^\circ.0019611$$

$$= 7''.059$$

the coefficient of sin. $(n'\,t - n\,t + \epsilon' - \epsilon)$.

In order to find the coefficient of sin. $2\,(n'\,t - n\,t + \epsilon' - \epsilon)$

$$\frac{n}{n - n'}\,a\,\Gamma = -\,\frac{n}{n - n'} \, . \, \rho' \times .02819 = -\ .0059198$$

therefore twice the above quantity = − .0118396

Again,

$$a^2\,\frac{d\,\Gamma}{d\,a} = -\,a^2 . \frac{d\,C}{d\,a} = -\,\rho'^2 \times .297995 = -\ .01105$$

$$\therefore\ a^2 \, . \, \frac{d\,\Gamma}{d\,a} + \frac{2\,n}{n - n'}\,a\,\Gamma \ldots\ldots = -\ .0228896$$

---

* log. S = 8.5243701

log. $\frac{n}{n - n'}$, = .0382490

log. arc (= rad.) = 1.7581226

.3207417

log. m'...... = 3.0282458

7.2924959 the logarithm of .0019611.

and, since $\dfrac{2N^2}{4\,.\,(n-n')^2-N^2}=-\dfrac{2n^2}{n'^2}\times\dfrac{1}{\left(\dfrac{3n}{n'}-2\right)\left(\dfrac{n}{n'}-2\right)}$, nearly,

$$\frac{2N^2}{4(n-n')^2-N^2}\left(a^2.\frac{d\Gamma}{da}+\frac{2n}{n-n'}a\Gamma\right) = -.019443$$

$$\text{but}\ \frac{n}{n-n'}\,.\,a\Gamma\ldots\ldots\ldots\ldots\ldots\ldots = -.005919$$

$$\therefore (S)\ \text{sum of two last lines}\ldots\ldots\ldots\ldots = -.024362$$

$$\therefore \frac{m'n}{n-n'}\times\frac{S}{2}\ldots\ldots\ldots\ldots\ldots\ldots = -0^{\circ}.006979$$

$$= -2''.51.$$

The numerical value of the coefficient of sin. $3(n't-nt+\epsilon'-\epsilon)$ is less than a second, being $0''.17$: and that of the coefficient of sin. $4(n't-nt+\epsilon'-\epsilon)$ is $0''.017$.

Hence the correction to the Earth's longitude arising from the perturbation of Jupiter, is

$$\begin{aligned}&7''.059\ \text{sin.}\,(n't-nt+\epsilon'-\epsilon)\\ &-2''.51\,.\,\text{sin.}\,2(n't-nt+\epsilon'-\epsilon)\\ &-\&\text{c.}\end{aligned}$$

which may be thus expressed,

$$\begin{aligned}&7''.059\,.\,\text{sin.}\,(♃-⊙)\\ &-2''.51\,.\,\text{sin.}\,2(♃-⊙),\end{aligned}$$

and this result agrees, very nearly, with that which Clairaut (*Mem. Acad.* 1754. p. 544.) has obtained by means of the differential equations of p. 95. and by an use of them similar to that which has already been made in Chapters VIII, IX, &c.

Precisely after the manner of the preceding computation, and by similar formulæ, we may compute the inequalities in the Earth's longitude arising from the actions of Mars and of Venus. The latter will be

$$5''.29\,.\,\text{sin.}\,(♀-⊙)-6''.1\ \text{sin.}\,2(♀-⊙),$$

the former

$$0''.4\ \text{sin.}\,(♂-⊙)+3''.5\ \text{sin.}\,2(♂-⊙),$$

which are the principal terms of the inequalities that are independent of the eccentricity.

Mercury, Saturn, and the Georgium Sidus affect but slightly the Earth's motion: so that the principal inequality in the Earth's longitude that arises from the perturbations of the Moon (see Chapters VI. XVI.), and of the planets (see pp. 271, 310.) may nearly be represented by the following formula *.

$$\begin{aligned}\delta v = {} & 8''.9 \,.\, \sin.(\text{☽} - \text{☉}) \\ & + 7''.059 \,.\, \sin.(\text{♃} - \text{☉}) - 2''.51 \,.\, \sin.\, 2(\text{♃} - \text{☉}) \\ & + 5''.29 \,.\, \sin.(\text{♀} - \text{☉}) - 6''.1 \sin.\, 2(\text{♀} - \text{☉}) \\ & + 0''.4 \,.\, \sin.(\text{♂} - \text{☉}) + 3''.5 \,.\, \sin.\, 2(\text{♂} - \text{☉}).\end{aligned}$$

The terms which are independent of the eccentricity and which depend on the angular distance of the disturbed and disturbing bodies, belong to an inequality, or expound an *equation*, analogous, as it has been already remarked in p. 307. to the Lunar Variation.

Now, of the terms expounding this latter equation, the second, that which has for its argument twice the angular distance of the Sun and Moon, is (see p. 219.) by far the greatest. But, (so little has analogy to do in these enquiries) no rule can thence be drawn relative to *planetary variations:* for, as we see, (p. 310. l. 12.) when Jupiter is the disturbing body, it is the coefficient of the term depending on ♃ — ☉ which is the largest.

In the preceding pages the inequality in longitude has been computed: for, that (if we regard the formation of Astronomical Tables, and the reflected evidence of the truth of the law of gravity, which their agreement with observation affords) is of more importance than the inequality in parallax, or the perturbation of the radius vector. But, in fact, this latter was necessarily found in investigating the former: for, see pp. 268. $r\,\delta r$ is one of the terms of the expression for δv.

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* The perturbations of the Earth, from the action of the Moon and the planets, form the subject of a most admirable Memoir of Clairaut's, entitled, *'sur l'Orbite apparente du Soleil,'* and inserted in the *Memoirs of the Academy,* for 1754. Almost all the difficulties that occur in the planetary theory (and the case contains them) are there met and overcome by Clairaut; amongst these are, the computation of *A, B,* &c. when ρ', in the case of Venus disturbing the Earth, is not a small fraction: again, the determination of the masses of the Moon and Venus: for except these latter quantities should be known [and they cannot be determined by the ordinary methods (see p. 61.)] the perturbations caused by the Moon and Venus would be altogether uncertain.

Now,

$$\frac{r\,\delta r}{a^2} = \frac{\delta r}{a} \cdot \frac{r}{a}$$

$$= \frac{\delta r}{a}\left[1 + e \cdot \cos.\,(n t + \epsilon - \pi) + \&c.\right]$$

therefore, in the research of inequalities independent of the eccentricity, we may assume $\frac{r\,\delta r}{a^2} = \frac{\delta r}{a}$, and, accordingly, (see p. 307.)

$$\frac{\delta r}{a} = m' \left\{ \begin{array}{l} \dfrac{n^2}{(n-n')^2 - N^2}\left(a^2 \cdot \dfrac{d\,\mathrm{B}}{d\,a} + \dfrac{2\,n}{(n-n')}\,a\,\mathrm{B}\right) \cos.\,\omega \\ + \dfrac{n^2}{4\,(n-n')^2 - N^2}\left(a^2 \cdot \dfrac{d\Gamma}{da} + \dfrac{2\,n}{n - n'}\,a\,\Gamma\right) \cos.\,2\omega \\ + \&c. \end{array} \right\}$$

= [when Jupiter is the body disturbing the Earth, (see pp. 308, 310.)]

$$.00001594 \cdot \cos.\,(\text{♃} - \odot)$$
$$- .000009109 \cdot \cos.\,2\,(\text{♃} - \odot).$$

The inequalities dependent on the eccentricities, are, in the greater number of cases, less in quantity than the preceding inequalities. The greatest of the inequalities (dependent on the first power of the eccentricities), which Jupiter's action causes in the Earth's longitude isabout two seconds and an half. But, although their quantity is small, the process of deducing and computing them is intricate and tedious: and even those inequalities, which are too small to be retained, cannot be rejected by any examination that is much short of actual computation.

The arguments of the terms constituting the equation that is analogous to the Lunar Variation, have this peculiarity of condition: namely, that the numbers multiplying $n' t$, $n t$ in the same term are always equal; the arguments are $n' t - n t + \epsilon' - \epsilon$,

$2n't - 2nt + 2\epsilon' - 2\epsilon$, and generally, $p\,.\,(n't - nt + \epsilon' - \epsilon)$. In the terms dependent on the first powers of the eccentricities, the numbers multiplying $n't$, nt (including 0 amongst the numbers) always differ by 1 : and, see p. 279, are generally expressed by

$$p\,(n't - nt + \epsilon' - \epsilon) + nt + \epsilon - \pi,$$

$$\text{and } p\,(n't - nt + \epsilon' - \epsilon) + n't + \epsilon' - \pi' :$$

Now, of the several arguments comprehended under the above two formulæ, that of which we have spoken at p. 312. l. 18. is $n't + \epsilon' - \pi'$, which arises on making $p = 0$ in the latter of the two formulæ. The coefficient of this we will now deduce.

Since the terms involving the eccentricity are to be taken account of, the form of the equation of p. 258, must be slightly altered : since, in the term $\frac{\mu r \delta r}{r^3}$, we must substitute, instead of $\frac{1}{r^3}$, its elliptical value, namely,

$$\frac{1}{a^3.[1 - e.\cos.(nt + \epsilon - \pi) + \&c.]^3} = \frac{1}{a^3}[1 + 3e.\cos.(nt + \epsilon - \pi) + \&c.]$$

in which case the equation (when terms, involving higher powers of the eccentricities than the first, are excluded) will become

$$0 = d^2.\left(\frac{r\delta r}{a^2.dt^2}\right) + n^2.r\delta r + 3n^2 a.\delta r\,[e\,.\cos.(nt + \epsilon - \pi)]$$

$$+ 2\int dR + r.\frac{dR}{dr},$$

The first step in the process towards integration ought to consist in finding δr from the integration of the more simple form of the equation; and, as it is evident, this form would result

$$\frac{r\delta r}{a^2} = m'(F + G.\cos.\omega + H.\cos.2\omega + \&c.)$$

and consequently, since

$$\frac{r\delta r}{a^2} = \frac{\delta r}{a}\cdot\frac{r}{a} = \frac{\delta r}{a}(1 - e\cos.nt + \epsilon - \pi),$$

$$\frac{\delta r}{a} = m' \left\{ \begin{array}{l} F + Fe.\cos.(nt + \epsilon - \pi) \\ + \frac{Ge}{2}.\cos.(n't + \epsilon' + \pi) + \frac{Ge}{2}\cos.(n't - 2nt + \epsilon' - 2\epsilon + \pi) \\ + \&c. \end{array} \right\}$$

Now, such a value must be substituted in the third term of the preceding equation, (p. 313. l. 19.) when the enquiry is concerning other terms than those which involve e' sin. $(n't + \epsilon' - \pi')$: but terms of this latter description can never enter into that equation by the effect of the value of δr in the third term: they must therefore enter into the value of $\frac{r\delta r}{a^2}$ (that value which results from integrating the equation of p. 313. l. 19.) from the value of $r\frac{dR}{dr} + 2\int dR$; and, therefore, in fact, from R containing such terms: and see p. 279. R does contain such a term, namely,

$$\frac{m'}{2}a'.\frac{dA}{da'}e'\cos.(n't + \epsilon' - \pi').$$

But, with reference to such term, dR, and, consequently, $2\int dR = 0$; and

$$\frac{rdR}{dr} = a.\frac{dR}{da} = \frac{m'}{2}.\frac{d^2A}{da.da'}e'.\cos.(n't + \epsilon' - \pi').$$

The corresponding term in $\frac{r\delta r}{a^2}$ would be the preceding divided by $n'^2 - n^2$. But in the value of δv, which we are seeking, it would disappear, since the symbol d in $d(r\delta r)$ refers solely to the attracted body (its ordinates, mean motion, &c.) $dr.\delta r$ may also be excluded, since dr involving e, the term, dependent on the argument $n't + \epsilon' - \pi'$, in $dr\delta r$, would involve ee'.

Hence, the value of δv is reduced to this (see pp. 269.)

$$\delta v = 2a\int r.\frac{dR}{dr}ndt$$

$$= nm'a^2a'.\frac{d^2A}{da.da'}e'\int\cos.(n't + \epsilon' - \pi').dt$$

$$= \frac{nm'a'}{n'}a^2.\frac{d^2A}{da.da'}e'.\sin.(n't. + \epsilon' - \pi');$$

but, from the formulæ of p. 298.

$$a^2 a'.\frac{d^2 A}{d\,a.d\,a'} = -2\,a^2.\frac{d\,A}{d\,a} + a^3.\frac{d^2 A}{d\,a^2}$$

$$* = -\frac{2\,a^2}{a'^2} \times .200586 - \frac{a^3}{a^3} \times 1.132355;$$

$$\therefore\ \delta v = -.000070034\,.\sin.(n't + \epsilon' - \pi')$$

$$= -2''.5\,.\sin.(n't + \epsilon' - \pi').$$

By similar but longer processes, the other inequalities dependent on the first powers of the eccentricities may be computed. If the argument, instead of solely involving $n't$ (which is the special cause of the abridgment of the computation), had been $n't - 2nt +$ &c., or $3n't - 2nt$, &c. dR and $\int dR$ would not have been equal nothing.

The argument $n't + \epsilon' - \pi'$ is under the condition (see p. 313.) that all arguments are dependent on the first power of the eccentricities: namely, the excess of the multiples of $n't$ (which is 1) above that of nt (which may be considered 0) is equal to 1, the index of the first power: and there is in the value of R (see p. 279.) another similar term under the same conditions, namely,

---

* log. 2 = .3010300 .... log. $\frac{a^3}{a'^3}$ = 7.8516979

log. $\frac{a^2}{a'^2}$ = log. ρ'^2 = 8.5676996

log. 2005886 = 9.3023006.......log. 1.132355 = 0.0539825

8.1710302 .... 7.9056804

No. = .014826......................No. = .0080479 (a)

(a) .0080479

.0228749, its l. = 8.3593421

e' = .0480767, l. e' = 8.6819347

log. arc. (= rad.) = 1.7581226

(see p. 308.) log. $\frac{n}{n'}$ 1.0741513

19.8435517

log. m' = 3.0282458

6.8453059 = log. 000070034.

$$\frac{m'}{2} \cdot \frac{dA}{da} e \cdot \cos. (nt + \epsilon - \pi),$$

which merits some attention.

With reference solely to this term

$$r\frac{dR}{dr} + 2\int dR =$$

$$\frac{m'e}{2}\left(a \cdot \frac{d^2 A}{da^2} + 2 \cdot \frac{dA}{da}\right) \cos. (nt + \epsilon - \pi);$$

consequently, the integration of the equation, by which $r\delta r$ is determined, would introduce (see pp. 97, 259.) into its value a term, having the arc or time *without* the sign; in fact, a term such as

$$Mnt \cdot \sin. (nt + \epsilon - \pi),$$

(see Chap. VIII.)

Such terms, as the above, existing in the values of the radius vector and longitude, would, by increasing with the time, materially alter the elements of the orbit. But, as it has been already explained, they are introduced by that method of approximation, which is imperfect, but which we are obliged to employ for the purpose of nearly integrating the differential equations. The values of r, r', v, v', for instance, which we employ (see p. 274.) for the purpose of deducing R, are their elliptical values, in which the eccentricities and perihelia are without change: conditions which do not take place in a disturbed system. We have already seen (see Chapter VIII.) how, in some cases, the faulty expressions for the radius, &c. may be amended. But a source of enquiry altogether new would be opened, if, beginning the investigation from such a term as $Mnt \cdot \sin. (nt + \epsilon - \pi)$, we sought, by reverse steps, the *periodical terms*, or *transcendental expressions* by the development of which it was produced. For, α being a very small quantity, and $\sin. \alpha t = \alpha t - \frac{(\alpha t)^3}{1.2.3} + \frac{(\alpha t)^5}{1.2.3.4.5} - \&c.$ $\sin. \alpha t = \alpha t$ nearly, when t is such that αt still remains small: and, under this restriction, the numerical value of v (supposing the longitude to be sought), would, in a specific instance, not be altered by using αt as the value of sin. αt. But such an use or substitution would altogether alter the nature of the general solution, and would affect it with *secular equations*. If then αt should, after the manner described, be introduced, the new enquiry, which we have spoken of, would be to find sin. αt.

We will attempt to describe slightly the grounds of this enquiry.

The elliptical value of v (see p. 274.) is

$$v = nt + \epsilon + 2e \,.\, \sin.(nt + \epsilon - \pi) + \&c.$$
$$= nt + \epsilon + 2k \,.\, \sin.(nt + \epsilon) - 2l \cos.(nt + \epsilon) + \&c.$$

by making

$$k = e \,.\, \cos.\,\pi, \quad l = e \sin.\,\pi.$$

The inequality (δv) produced in the elliptical value by the disturbing force will consist of a great variety of terms, all multiplied by m' the mass of the disturbing body. Amongst these (if we examine the value of R in p. 279. and the expression for δv in p. 268.) there will be terms of this form

$$m'(lK + l'L)\,nt \,.\, \sin.(nt + \epsilon),$$
$$m'(kK + k'L)\,nt \,.\, \cos.(nt + \epsilon),$$

l' and k' being terms similar to l, k, that is,

$$k' = e' \,.\, \cos.\,\pi', \quad l' = e' \,.\, \sin.\,\pi'.$$

That part (v) therefore, of the longitude which depends on the sines and cosines of the angle $nt + \epsilon$, may be thus expressed,

$$(v) = [2k + m'(lK + l'L)\,nt] \sin.(nt + \epsilon)$$
$$+ [2l + m'(kK + k'L)\,nt] \cos.(nt + \epsilon).$$

Now k, l, &c. are (see l. 13.) functions of the eccentricity (e) and perihelion (π). In the elliptical values of r and v, (which are used in the first approximations) e and π are constant; therefore k and l are: but when the disturbing force acts they become variable. We may consider then the factors of sin. $(nt + \epsilon)$, cos. $(nt + \epsilon)$ in the above expressions as the two first terms of *developed* expressions for k and l; k and l varying with the time. If therefore k and l were the values of e cos. π,

$e \cdot \sin \pi$, at a certain epoch, when t, then denoting the commencement of time, was 0, we should have (see *Anal. Calc.* p. 83.)

$$2\left(k + \frac{dK}{dt}.t\right),$$

$$2\left(l + \frac{dl}{dt}.t\right),$$

to represent the two first terms of the developed functions of e and π, and by comparison of terms, we have

$$\frac{dk}{n\,dt} = \frac{m'}{2}\,(l\,K + l'\,L),$$

$$\frac{dl}{n\,dt} = \frac{m'}{2}\,(k\,K + k'\,L),$$

and from these equations are to be derived, by integration, those *transcendental** expressions, or functions, which, partially developed, produce the arc or time *without* the sign, and apparently render faulty the expressions for the radius vector and longitude.

Since $k = e \cdot \cos. \pi$, and $l = e \cdot \sin. \pi$,

$$dk = de \cos. \pi - e\,d\pi \,.\, \sin. \pi,$$

$$dl = de \,.\, \sin. \pi + e \,.\, d\pi \,.\, \cos. \pi.$$

The above investigation then leads to the finding of the variations of eccentricity and perihelion, which are two of the elements of the orbit: and a similar investigation (taking account of those terms which are introduced by the inclination of the planes of the orbits) would lead to the *variations* of the nodes and inclination, which are two other elements †. But both in the one, and the other case, we should be led by a reverse process: by a process, in fact, as little simple and obvious as any that can be imagined, and which must necessarily belong to a refined state of analytical science.

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* $k = .\,A \sin. (ft+g) + B \sin. (f'\,t + g')$ would be a transcendental expression.

† The subject of the *secular inequalities* (for such it is) was treated of, after the manner alluded to in the text, by Laplace in the *Mem. Acad.* 1785; and subsequently, with greater refinement of calculation, but much less perspicuity, in his *Mecanique Celeste.*

In a subsequent Chapter of this Work, the variations of the elements of a planet's orbit, or its *secular inequalities* will be treated of by a more direct method than the one that has been just described. We will now consider whether there are any *periodical* inequalities other than those already investigated, that claim our attention. We shall find such in the theory of Jupiter and Saturn.

The perturbations of these planets, require, like those of Venus and the Earth, that special or peculiar computation which has been described in pp. 286, &c.

$$\text{for, } \rho' = \frac{\text{rad. of ♃'s orbit}}{\text{rad. of ♄'s orbit}} = .54531725.$$

It would seem then that no cases could be more alike than the preceding; and that the solution of the *Problem of the three Bodies*, for Venus the Earth and Sun, would be, virtually and in substance, the just solution, when Jupiter, Saturn and the Sun should be the three bodies.

But here, as frequently in intricate investigations, it happens that general views and analogies are altogether fallacious. The theory of the perturbations of Jupiter and Saturn contains very distinct peculiarities. It differs, in certain respects, not only from that of the perturbations of Venus and the Earth, but from every other planetary theory. The only points of resemblance to it are to be found in the system of Jupiter's Satellites. But we will proceed to explain, without farther preamble, in what the peculiarities above alluded to consist.

CHAP. XIX.

On certain Inequalities of Jupiter and Saturn, which depend on the near Commensurability of their Mean Motions. Five times Saturn's Mean Motion nearly equal to twice Jupiter's. The peculiar Inequalities of Jupiter and Saturn expounded by Terms involving the Cubes of the Eccentricities. The Cause of their magnitude. Connexion, in the same Term, between the Power of the Eccentricity and the Form of the Argument. Expressions for the Retardation of Saturn, and the corresponding Acceleration of Jupiter. Agreement of the Results of Computation and Observation. Period of the Inequality. A similar Inequality in the Motion of Mercury, &c. &c.

The condition that renders singular the case of the mutual perturbations of Jupiter and Saturn, is the numerical relation that subsists between their mean distances; which is such that the mean motions of the two planets are to one another almost in a definite proportion.

If we examine the value of R, and the forms of the integrals by which the longitude and parallax are expressed, we shall easily perceive what kind of peculiarity of result must ensue, if n the mean motion of the disturbed should be to n' the mean motion of the disturbing planet nearly as number is to number.

Take the most simple case: suppose n to be to n' nearly as 1 to 2: now, one of the terms of R (see p. 279.) is of the form

$$P e . \cos. (2 n t - n' t + 2 \epsilon - \epsilon'),$$

and, in consequence of this term and corresponding to it, there will be introduced into $d R$ a term such as

$$-\,2\,P\,n\,e.d\,t.\sin.(2\,n\,t - n'\,t + 2\,\epsilon - \epsilon'),$$

and, accordingly, into the value of δv, and by virtue of the term $3\,a\!\int\!\!\int n\,dt.d\,R$ which it contains, this term

$$\frac{6\,P\,n^2\,a\,e}{(2\,n-n')^2}\sin.(2\,n\,t - n'\,t + 2\,\epsilon - \epsilon').$$

Now $2\,n - n'$ is by supposition very small: the coefficient, therefore, of the above term would receive from the divisor $(2\,n - n')^2$ (by as much indeed as that divisor can confer) considerable magnitude, and the term, in its resulting value, notwithstanding the minuteness of P and e, might expound an equation of considerable moment.

The magnitude of the equation is not the sole consequence of the minuteness of $2\,n - n'$. The period of the equation, as it is plain from pp. 235, 236: would be increased by it, and become greater the smaller $2\,n - n'$ should be.

The case we have put is altogether hypothetical: amongst the planets there are no two whose mean motions are either as 2 to 1, or nearly so*.

But if n' should be nearly to n either as 3 to 2, or, as 3 to 1, or, as 5 to 2, or, as 4 to 1, or as, &c., or generally as i' to i, there would arise, in any one of these cases, an equation of some magnitude and with a long period: the length of the latter depending on the minuteness of $i'\,n' - i\,n$: the magnitude of the former depending partly on that condition and on other conditions.

The first point is easily made out; if we revert to the note of p. 235, it will appear that the *period* of the equation, or that interval of time in which it will pass through all its degrees and *affections* of magnitude, will be the larger the smaller $i'\,n' - i\,n$ is: but the magnitude of the coefficient (which is the greatest value of the equation) must depend partly on that of e, or, that being given, on the power of e which it involves.

This brings us to the very jet of the business: the term in the differential equation may involve e^2, or e^3, or, e^4, and, in consequence thereof, may be extremely small: but the corresponding term in the integrated equation, may, by having received a small divisor, become of some value: in other words, a very small modification of the disturbing force, may, by the duration of its agency, or by the accumulation of its effects, sensibly affect the disturbed planet's place.

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* The mean motions of the first and second, of the second and third Satellites of Jupiter, are, however, in that proportion.

The terms *likely* to be neglected by the computist would be those involving the squares and cubes of the eccentricity. 'Nous pouvons (says Euler in an ineffectual Essay to explain the irregularities of Jupiter and Saturn) *hardiment* negliger les termes qui renferment le quarré et les plus hautes puissances de l'eccentricité.' The cube of the eccentricity of Jupiter's orbit is .00011183: the terms, therefore, that involve both this fraction and the fraction expounding the disturbing force must be extremely small in the differential equation. They are the very terms, however, as we shall soon see, that require, in the theory of Jupiter and Saturn, particular consideration.

The very minute modifications of the disturbing force, expounded by such terms, can produce effects only in one way; that is, by the great duration of their agency: in other words, their periods must be very large: if, therefore,

$$P e . \cos . (i' n' t - i n t + i' \epsilon' - i \epsilon),$$

should represent one of the above-mentioned terms, it would follow (see p. 235.) that

$$i' n' - i n,$$

must be a very small quantity.

The terms then in the differential equation that are extremely small from involving e^3, and the quantity expounding the disturbing force, may become of moment from receiving by integration (which is the scientific summation of small terms) divisors such as $i' n' - i n$, or $(i' n' - i n)^2$. But there is no necessary connexion whatever between the minuteness of $i' n' - i n$, and that of e^3.

If we wish to know, antecedently to actual computation, whether the mean motions n' and n are so related, that, i' and i being two integers, $i' n' - i n$ can be either nothing or nearly so:

we must examine the numbers expressing the mean motions and make trial with them. Now in the case of Jupiter and Saturn,

n' (♄'s mean annual motion) $= 43996''.72$
n (♃'s mean annual motion) $= 109256''.23$

if therefore we take $i' = 5$, and $i = 2$, we shall have

$$5n' - 2n = 1471''.14,$$

a small quantity relatively to n or n': and these integers 5 and 2, are the only ones, having a difference equal to 3, that make $i'n' - in$ a small quantity. In the other planets, whatever be the two selected, there are no two integers i' and i (having a difference either 1, 2, 3 or 4) that make $i'n' - in$ a quantity equally small with the preceding.

But we have not yet shewn what the terms are that have the argument $5n' - 2n$: the fact is, such terms involve the cube of the eccentricity, and on that account are extremely small: but they expound a modification of the disturbing force, the agency of which, either continually accelerating, or continually retarding the body's motion, must endure for a very long time: for, since $5n' - 2n = 1471''$, the *whole period* of its action (see p. 235.) is about 900 years.

Having thus ascertained, by antecedent considerations, the existence of a very small inequality of a very long period, let us consider in what manner it would affect the phenomena of observation and their determination.

The mean motion of a planet (see *Astron.* p. 263.) is determined by observing the planet in two similar oppositions, and then by dividing the interval of time by the number of revolutions.

Now, an inequality, such as has been described, acting almost by insensible degrees, and for a great length of time, would affect the determination of the mean motion. Its effects would be blended with it; and without the aid of theory it would be impossible to disengage them. For, if the annual effect of the inequality should not exceed a few seconds, and its period should be 900 years, no comparison of observations, made during an

Astronomer's life time, could possibly point out that *configuration* of the Sun, Jupiter, and Saturn (supposing these latter to be the mutually disturbing bodies) on which the inequality depended.

Suppose in determining the mean motion, the inequality, about the time of the first observed opposition (see *Astronomy*, p. 263.), was at the beginning of its period and began to *augment* the body's motion; then, the mean motion determined by dividing the difference of longitudes between the first and a second opposition (made at a less interval than 450 years) by the number of revolutions, would be too large. If there were three intervals, of 120 years each, between four similar oppositions, the mean motion determined from the comparison of the two first oppositions would be less than the mean motion determined from the second and third opposition, and still less than the result from the third and fourth opposition. The inference from such observations would be that the *mean motion was accelerated:* and a modification of the disturbing force, undergoing changes only very gradual and minute, would, for short periods, have all the effects of an uniformly accelerating or retarding force. There would be, in the case we are considering, an acceleration of the mean motion precisely similar to Galileo's Acceleration of Spaces. The same formulæ would suit both cases: if P represented the disturbing force, t the time, and n the mean motion; then the difference of the body's longitude after a lapse of time equal to t would be (independently of the ascertained periodical inequalities),

$$n t \pm P t^2,$$

and $P t^2$, would, in such a case, expound a *secular equation*, one really so, and increasing with the time.

If, instead of accelerating, the disturbing force should retard the body's motion (as it would do at another part of the period of its action) the mean motion determined by the preceding methods (see ll. 10, &c.) would be less and less, and would appear to be *retarded.* And, in fact, it was a *retardation* of Saturn's mean motion which was first noted by Flamstead.

We have now shewn, taking our departure from the formulæ of calculation, that a peculiarity of result, theoretically possible for any two planets, will actually take place in the theory of Jupiter and Saturn. We will now proceed on the reverse course, and examine whether observation presented any *anomalous* phenomenon of which such *peculiarity* of result, as that we have spoken of, might prove to be the explanation.

The mean annual motion of Saturn, like that of any other superior planet, is determined, as it has been already stated (p. 323) by comparing two similar oppositions, and by dividing the difference of his longitudes by the interval of years. The greater the interval, the greater, *ceteris paribus*, will be the accuracy of the result; since a small error distributed over a great number of years would be nearly absorbed and become insensible. Now, in determining Saturn's mean motion, a recorded opposition that happened 228 years before Christ, was compared with an opposition observed in Feb. 26, 1714. The elapsed interval was $1943^{y}.\ 218^{d}.\ 1^{h}.\ 51^{m}$: the number of revolutions 66. Saturn's periodic time then was $29^{y}.\ 162^{d}.\ 4^{h}.\ 27^{m}$, and his mean annual motion $12^{\circ}.\ 13'.\ 35''.\ 14'''$.

This determination of the mean motion must, for the reasons just alledged, be a very exact one. We shall hereafter see that it cannot be erroneous to the amount of three seconds, even if the inequality which we have spoken of, should be supposed to operate with its greatest effect.

But *modern* observations (as they may be called with reference to those above cited) gave a different result for Saturn's mean motion. The oppositions, for instance, of the years 1594, 1595, 1596, 1597, compared with those of 1713, 1714, 1715, 1716, made Saturn's period *longer* than $29^{y}.\ 162^{d}.\ 4^{h}.\ 27^{m}$ (see l. 19.). In other words, his mean motion during the preceding intervals of 120 years might be said to be *retarded*.

Now this *retardation* was an *anomalous phenomenon*. The law of constancy in their mean motions, which the other planets observed, was departed from. If Saturn's mean motion grew less and less, his mean distance would become greater and greater; he would be in a perpetual state of recess from the centre of the system, and the *stability* of the planetary system, as it is called, could neither exist as a fact of observation, nor as a result of theory.

Lalande and other Astronomers said that Saturn's mean motion was subject to a *secular* equation: they represented the mean motion by

$$nt \pm P.t^2,$$

and they determined, by the comparison of observations, the numerical value of P the coefficient of the secular equation.

But an *empirical* equation, supplied for the purpose of remedying the defect of Saturn's Tables, was no explanation of the anomalous phenomenon. The difficulty to be surmounted remained the same in substance as before. Was the existence of a secular equation compatible with the system of universal gravitation; or could one exist peculiarly for the theory of Jupiter and Saturn?

The retardation of Saturn's mean motion, and the *acceleration* of Jupiter's were first noted by Flamstead, who in 1682 observed a conjunction of these planets. Halley, the contemporary of Newton, found also the Tables of Jupiter and Saturn to be incorrect. But the great founder of Physical Astronomy, whether he considered the anomalous phenomenon of Saturn's retardation as not sufficiently ascertained, or whether he wanted leisure for the research, has no where adverted to that phenomenon. He certainly did not view it as forming an exception to his system: for, in speaking of the perturbations of the planets, he merely says that the action of Jupiter is a thing not entirely to be passed over 'Actio quidem Jovis in Saturnum non omnino contemnenda est.' On the subject of these two planets he does not notice that peculiarity of their theory, which for a time seemed to form an exception to his system, but which afterwards became one of its strongest confirmations.

But the mathematicians who succeeded Newton and followed his system, were greatly embarrassed with the *retardation* of

Saturn's mean motion. As a fact of observatión it was anomalous; and theory, so far from exhibiting it as a result of calculation, gave a result directly the opposite. For, in the year 1774, Lagrange, by means of a remarkable theorem *, proved the *invariability* of the mean distances of the planets. If the mean distances remained the same, or were subject (as is the case) to periodical inequalities, the mean motions, if Newton's theory were true (see p. 29.) must be so also. They could admit neither of *secular* retardation nor acceleration.

These considerations, then, begat a strong belief that the retardation of Saturn's mean motion was a phenomenon explicable on Newton's principles; and, with a view of calling the attention of mathematicians particularly to this point, the Academy of Sciences of Paris proposed as the subject of their prize for the year 1748, *The Theory of Jupiter and Saturn.* This produced two fruitless, although in other respects excellent, disquisitions from Euler and Lagrange; which obtained the prize, but left the difficulty as they found it †.

But the subsequent investigations of Laplace had better success. That excellent mathematician having shewn, as Lagrange had, that the mean distances of the planets, notwithstanding their mutual perturbations, were subject only to *periodical* inequalities, proceeded to prove, that if the *retardation* of Saturn arose from Jupiter's action, the action of Saturn ought to cause an *acceleration* in Jupiter's mean motion, and in a given proportion to the retardation.

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* If a be the mean distance,

$$d\,a = -\frac{2a^2}{\mu}\,d\,R.$$

† Or, bien que J'avoue franchement que Je ne suis pas en etat d'expliquer parfaitement toutes les irregularités qui se trouvent dans le mouvement de Saturne, je crois pourtant pouvoir pretendre aux prix que l'academie propose, et méme avec plus de droit que ceux qui ne se sont pas appercus, &c. Euler, tom. VI. *Prix Acad. des Sciences.*

By a new scrutiny of observations, Laplace found that the corresponding retardations and accelerations of Saturn's and Jupiter's motion, were in that proportion which theory gave. It was probable then that they arose from the mutual perturbations of the two planets, and that the principle and law of gravity were sufficient to account for them.

The theorem from which Laplace inferred an acceleration in Jupiter's motion corresponding to a retardation in Saturn's was this: m, m', being the masses, and a, a' the mean distances of the two planets, then

$$\frac{m}{a} + \frac{m'}{a'} = f,$$

f being a constant quantity.

Take the differential of the above equation, then

$$\frac{m\,d\,a}{a^2} = -\frac{m'\,d\,a'}{a'^2};$$

$$\text{but, } n^2 = \frac{1}{a^3}, \quad n'^3 = \frac{1}{a'^3},$$

consequently,

$$d\,n = -\frac{3}{2}\cdot\frac{d\,a}{a^2}\cdot\frac{1}{\sqrt{a}},$$

$$d\,n' = -\frac{3}{2}\cdot\frac{d\,a'}{a'^2}\cdot\frac{1}{\sqrt{a'}};$$

whence,

$$\frac{d\,n}{d\,n'} = -\frac{m'}{m}\sqrt{\frac{a'}{a}}.$$

The variation ($d\,n$) therefore in Jupiter's mean motion was to the variation ($d\,n'$) in Saturn's as $-m'\sqrt{a'} : m\sqrt{a}$. If $d\,n$ denoted an acceleration, $d\,n'$, by reason of the negative sign, would denote a retardation: and *vice versâ:* and since

$$m = \frac{1}{1067.09},$$

$$m' = \frac{1}{3359.4},$$

$$a = 5.20279,$$

$$a' = 9.53877,$$

$$\log.\ \frac{1}{m} = 3.02820 \ldots\ldots\ldots\ldots \log.\ \frac{1}{m'} = 3.52626$$

$$\log.\ \surd a' = .98974 \ldots\ldots\ldots\ldots \log.\ \surd a = 0.85816$$

$$4.01794 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 4.38442$$

$$\therefore \text{No.} = 10422 \ldots\ldots\ldots\ldots\ldots\ldots \text{No.} = 24233,$$

$$\begin{aligned} dn : -dn' &:: \frac{\surd a'}{m} : \frac{\surd a}{m'} \\ &:: 10422 : 24233 \\ &:: 3 : 7, \text{ nearly,} \end{aligned}$$

which accorded with observations.

The theorem (see p 328.) from which Laplace deduced the above variations of the mean motions was deduced from the principle and law of gravity. That theorem, therefore, shewed the inequalities to be mutually produced. But the other theorem (see p. 327.) shewed that the inequalities could not be *secular;* and if not secular, then Saturn's *retardation* could not perpetually remain such, but would at length become an *acceleration;* and, if so, Jupiter's acceleration would, by virtue of the first theorem, be contemporaneously converted into a *retardation:* but such alterations constitute the character of a *periodical* inequality. A periodical inequality, however, of a very long period, would, during certain intervals of its action, appear like a *secular* inequality. If such an inequality then could be detected in the formulæ representing the longitudes of Jupiter and Saturn, it might serve to explain observed anomalies in their mean motions. It would really explain them if its computed quantity agreed with observation.

In the hope of detecting such an inequality, Laplace examined all the terms of the formulæ by which the solution of the problem of the three bodies is expressed: and he detected it amongst the terms that involve the cubes of the eccentricities. Euler's research, therefore, (see p. 322.) had been inevitably fruitless, since in its outset he had, without fear of error, neglected such terms.

In p. 321. it was observed that there was no connexion between the minuteness of e^3 which made the terms involving it in the differential equation very small, and the minuteness of $5n' - 2n$ which made the corresponding terms in the integral equation very great. The minuteness of $5n' - 2n$ depends on the conditions of the individual case, and is peculiar to that of Jupiter and Saturn. But between e^3 and the form $5n' - 2n$, and generally between the power of e and the *form* $i'n' - in$, inasmuch as that form depends on the difference $i' - i$, there is a connexion. This it is easy to see by an inspection of the value of R, and by considering the nature of its formation. The terms, for instance, that involve e, and e' have for their arguments

$$n't - 2nt + \epsilon' - 2\epsilon + \pi,$$
$$nt - 2n't + \epsilon - 2\epsilon' + \pi,$$
$$2n't - 3nt + 2\epsilon' - 3\epsilon + \pi,$$
&c.

or generally,

$$p(n't - nt + \epsilon' - \epsilon) + nt + \epsilon - \pi,$$
$$p(n't - nt + \epsilon' - \epsilon) + n't + \epsilon' - \pi',$$

it is impossible then, that the difference of the multipliers of $n't$ and nt, [since it must be either $p-(p-1)$, or $(p+1)-p$] can be greater than 1, which is the index both of e and e'.

In like manner it is easy to see that the difference of the integers multiplying $n't$ and nt in the arguments of the terms that involve e^2, ee' and e'^2 (quantities of two dimensions) can never exceed 2. It may be less and equal nothing: so that either

$$P.\cos.\,[p(n't - nt + \epsilon' - \epsilon) + 2nt + K],$$
$$\text{or, } Q.\cos.\,[p(n't - nt + \epsilon' - \epsilon) + L],$$

will generally represent such terms. Again, in the terms involving e^3, e^2e', $e'e^2$, or e'^3, the difference of the integers multiplying $n't$, nt in the arguments can never exceed 3 (the dimensions of e^3, e^2e', $e'e^2$, e'^3) it may be less and equal 1, so that either

$$P.\cos.\,[p(n't - nt + \epsilon' - \epsilon) + 3nt + K],$$
$$\text{or } Q.\cos.\,[p(n't - nt + \epsilon' - \epsilon) + nt + L].$$

will generally represent such terms. Now, the terms represented by the first of these latter terms, and when $p = 5$, are the only ones with which we have any concern: for then the argument of the terms is

$$5n't - 2nt + 5\epsilon' - 2\epsilon + K,$$

and $5n' - 2n$ (see p. 323.) is a very small quantity: which is not the case with $4n' - n$, $6n' - 3n$, &c. that arise on making p equal to 4, 6, &c.

The terms that involve the cubes of the eccentricity, are not the sole terms in the development of R that depend on the angle or argument,

$$5n't - 2nt + A:$$

there are, for instance, terms involving the fifth powers of e, and e', that depend on the same angle: but, although in the theory of Jupiter and Saturn it is necessary to consider even such terms, still they are by far less important, by reason of their greater minuteness, than the terms involving e^3, e^2e', &c.

The terms in the values of $\int dR$ and $\int\int n\,dt.dR$ (see p. 268.) corresponding to the terms in R that depend on the angle $5n't - 2nt + A$, become large in the theory of Jupiter and Saturn, because $5n' - 2n$ is a small quantity: but, if $5n' - 2n$ be nearly equal 0, $5n' - 3n = 5n' - 2n - n$ must nearly equal $-n$. Now the equation of p. 313. admits a peculiar integration. If

$$P \cos.\,[p(n't - nt + \epsilon' - \epsilon) + 2nt + 2\epsilon + A],$$

be a term in that differential equation,

$$\frac{P}{[p(n' - n) + 2n]^2 - n^2},$$

will be the coefficient of the corresponding term in the integral equation. If p, therefore, $= 5$, the coefficient will be

$$\frac{P}{(5n' - 3n)^2 - n^2},$$

which, therefore, may become considerable, since the denominator is a small quantity.

It is necessary then particularly to attend to those terms in R which involve e^2, e'^2, ee', and which depend on the angle $(5n't - 3nt + A)$: but the differential equation will contain other such terms besides those that R contains: for the equation, see p. 258. is

$$0 = \frac{d^2 . r\delta r}{dt^2} + \frac{\mu . r\delta r}{r^3} + 2\int dR + r.\left(\frac{dR}{dr}\right),$$

which, since (see pp. 29. 32.)

$$\frac{\mu}{a^3} = n^2,$$

and

$$r = a\left(1 + \frac{1}{2}e^2 - e.\cos.(nt + \epsilon - \pi) - \frac{e^2}{2}\cos.(2nt + 2\epsilon - 2\pi)\right),$$

becomes,

$$0 = d^2 . \frac{r\delta r}{dt^2} + n^2 . r\delta r$$

$$+ 3n^2 a\delta r\,[e.\cos.(nt + \epsilon - \pi) + e^2.\cos.(2nt + 2\epsilon - 2\pi)]$$

$$+ 2\int dR + r.\frac{dR}{dr},$$

now δr, (the result not being extended beyond the simple powers of e and e'), will be expressed by

$$F.\cos.p\,(n't - nt + \epsilon' - \epsilon)$$
$$+ Ge.\cos.[p\,(n't - nt + \epsilon' - \epsilon) + nt + \epsilon - \pi]$$
$$+ He'.\cos.[p\,(n't - nt + \epsilon' - \epsilon) + nt + \epsilon - \pi']$$

which combined, according to trigonometrical formulæ, with

$$e.\cos.(nt + \epsilon - \pi) + e^2.\cos.(2nt + 2\epsilon - 2\pi),$$

will produce, together with other terms, terms involving in their coefficients e^2, ee', and having for their arguments

$$p\,(n't - nt + \epsilon' - \epsilon) + 2nt + 2\epsilon - 2\pi,$$
$$p\,(n't - nt + \epsilon' - \epsilon) + 2nt + 2\epsilon - \pi - \pi',$$

and, on making $p = 5$, the arguments

$$5n't - 3nt + 5\epsilon' - 3\epsilon - 2\pi,$$
$$5n't - 3nt + 5\epsilon' - 3\epsilon - \pi - \pi'.$$

Now, the cosine of the first of these arcs

$$= \cos.(5n't - 3nt + 5\epsilon' - 3\epsilon)\cos.2\pi$$
$$+ \sin.(5n't - 3nt + 5\epsilon' - 3\epsilon)\sin.2\pi$$

of the second

$$\cos.(5n't - 3nt + 5\epsilon' - 3\epsilon)\cos.(\pi + \pi')$$
$$+ \sin.(5n't - 3nt + 5\epsilon' - 3\epsilon)\sin.(\pi + \pi'),$$

and, consequently, with reference solely to the terms that depend on the above arguments (ll. 8, 9.) the differential equation may be thus generally expressed :

$$\frac{d^2.r\delta r}{dt^2} + n^2.r\delta r + n^2a^2P\cos.(5n't - 3nt + 5\epsilon' - 3\epsilon)$$
$$+ n^2a^2.Q.\sin.(5n't - 3nt + 5\epsilon' - 3\epsilon)$$

whence,

$$\frac{r\delta r}{a^2} = \frac{n^2}{(5n' - 3n)^2 - n^2}\left\{\begin{matrix} P\cos.(5n't - 3nt + 5\epsilon' - 3\epsilon) \\ + Q\sin.(5n't - 3nt + 5\epsilon' - 3\epsilon) \end{matrix}\right\},$$

the coefficients of which terms may become considerable, since

$$(5n' - 3n)^2 - n^2 = (5n' - 2n)(5n' - 4n),$$

is a small quantity from the smallness of $5n' - 2n$.

With regard to certain terms involving e^3, e^2e', &c. that become large in the expression for $\int dR$: suppose

$$m'k.\cos.(5n't - 2nt + 5\epsilon' - 2\epsilon - g\pi - g'\pi'),$$

to be the general representative of these terms, then, making

$$k.\sin.(g\pi + g'\pi') = P,$$
$$k.\cos.(g\pi + g'\pi') = Q,$$

it equals

$$m'P.\sin.(5n't - 2nt + 5\epsilon' - 2\epsilon)$$
$$+ m'Q.\cos.(5n't - 2nt + 5\epsilon' - 2\epsilon),$$

and, $a\frac{dP}{da}$, $a\frac{dQ}{da}$ being the terms of P and Q corresponding to $r\frac{dR}{dr}$,

$$2\int dR + r.\frac{dR}{dr} =$$

$$-m'\left(\frac{4n}{5n'-2n}P + a.\frac{dP}{da}\right)\sin.\,(5n't - 2nt + 5\epsilon' - 2\epsilon)$$

$$-m'\left(\frac{4n}{5n'-2n}Q + a.\frac{dQ}{da}\right)\cos.\,(5n't - 2nt + 5\epsilon' - 2\epsilon),$$

in which expression, from the largeness of the factor $\frac{4n}{5n'-2n}$, we may neglect $a.\frac{dP}{da}$, and $a.\frac{dQ}{da}$. If, then, these terms be neglected, and the differential equation,

$$0 = \frac{d^2\, r\delta r}{dt^2} + n^2.r\delta r + \&c. + 2\int dR + r.\frac{dR}{dr},$$

be integrated solely with reference to the terms that remain, there will result

$$r\delta r = \frac{-4m'n}{(5n'-2n)\left[(5n'-2n)^2 - n^2\right]}\left\{\begin{array}{l} P\sin.\,(5n't - 2nt + 5\epsilon' - 2\epsilon) \\ +Q.\cos.\,(5n't - 2nt + 5\epsilon' - 2\epsilon)\end{array}\right\}$$

but (see p. 331.)

$$(5n' - 2n)^2 - n^2 = -n^2 \text{ (nearly) } = -\frac{1}{a^3},$$

accordingly,

$$\frac{r\delta r}{a^2} = \frac{4m'n}{5n'-2n}\left\{\begin{array}{l} aP.\sin.\,(5n't - 2nt + 5\epsilon' - 2\epsilon) \\ +aQ.\cos.\,(5n't - 2nt + 5\epsilon' - 2\epsilon)\end{array}\right\}.$$

The value of $\frac{r\delta r}{a^2}$, therefore, contains terms involving the squares of the eccentricity (see p. 333.) and dependent on the angle,

$$5n't - 3nt + 5\epsilon' - 3\epsilon,$$

and terms involving the cubes of the eccentricities and dependent on the angle

$$5n't - 2nt + 5\epsilon' - 2\epsilon,$$

both rendered considerable by the smallness of the divisor $5n' - 2n$. There will also be produced, by the combination of terms, in the value of $\frac{r\delta r}{a^2}$ another term with the same divisor $5n' - 2n$, but dependent on the angle $5n't - 4nt$.

We will now proceed to consider the inequality of longitude, which is the chief object of enquiry, and to attain which, as it is evident from the equation of p. 268. the preceding deduction of certain terms in $r\delta r$ is necessary. The value of δv, neglecting the denominator $\sqrt{(1-e^2)}$, is

$$\delta v = \frac{d(r\delta r) - dr.\delta r}{a^2 n dt} + 3a\int\int n dt.dR + 2\int n dt.a^2.\left(\frac{dR}{da}\right).$$

Now, the terms in $r\delta r$ which involve e^3, e^2e', $e'e^2$, e'^3, depend on the angle

$$p(n't - nt + \epsilon' - \epsilon) + 3nt + 3\epsilon + A:$$

but the present enquiry is solely concerning those terms which arise on making $p = 5$.

Now the terms in δv, corresponding to such terms in the value of $d(r\delta r)$ will all become very small by reason of the factor $5n' - 2n$. If, therefore, we neglect $d(r\delta r)$, we shall have

$$\delta v = -\frac{dr.\delta r}{a^2 n dt} + 3a\int\int n dt.dR + 2\int n dt.a^2\left(\frac{dR}{da}\right).$$

In order to find the value of the first term, suppose

$$\frac{rdr}{a^2} = aH.\sin.(5n't - 3nt + 5\epsilon' - 3\epsilon)$$
$$+ aG.\cos.(5n't - 3nt + 5\epsilon' - 3\epsilon),$$

H and G containing (see pp. 332. 334.) e^2, ee', or e'^2: and having $5n' - 2n$ for their divisor; then, since

$$\frac{r}{a} = 1 - e\cos.(nt + \epsilon - \pi) - \&c.$$

$$\frac{dr}{a}\cdot\frac{\delta r}{nadt} = \frac{1}{2} Gae \sin.(5n't - 2nt + 5\epsilon' - 2\epsilon - \pi)$$

$$- \frac{1}{2} Hae \cos.(5n't - 2nt + 5\epsilon' - 2\epsilon - \pi),$$

next, the terms in R dependent on the angle $5n't - 2nt$ being (see p. 333.) represented by

$$m'P \sin.(5n't - 2nt + 5\epsilon' - 2\epsilon)$$
$$+ m'Q \cos.(5n't - 2nt + 5\epsilon' - 2\epsilon),$$

$$3a\int\int ndt.dR =$$

$$\frac{6n^2m'}{(5n' - 2n)^2}\left\{\begin{array}{l} aP \cos.(5n't - 2nt + 5\epsilon' - 2\epsilon) \\ - aQ \sin.(5n't - 2nt + 5\epsilon' - 2\epsilon)\end{array}\right\},$$

lastly,

$$2\int ndt.a^2.\frac{dR}{da} =$$

$$-\frac{2m'n}{5n' - 2n}\left\{\begin{array}{l} a^2.\frac{dP}{da} \cos.(5n't - 2nt + 5\epsilon' - 2\epsilon) \\ - a^2.\frac{dQ}{da} \sin.(5n't - 2nt + 5\epsilon' - 2\epsilon)\end{array}\right\}$$

Hence,

$$\delta v =$$

$$\frac{6m'n^2}{(5n' - 2n)^2}\left\{\begin{array}{l} aP \cos.(5n't - 2nt + 5\epsilon' - 2\epsilon) \\ - aQ \sin.(5n't - 2nt + 5\epsilon' - 2\epsilon)\end{array}\right\}$$

$$-\frac{2m'n}{5n' - 2n}.\left\{\begin{array}{l} a^2.\frac{dP}{da} \cos.(5n't - 2nt + 5\epsilon' - 2\epsilon) \\ - a^2.\frac{dQ}{da}.\sin.(5n't - 2nt + 5\epsilon' - 2\epsilon)\end{array}\right\}$$

$$+ \frac{Hae}{2}.\cos.(5n't - 2nt + 5\epsilon' - 2\epsilon - \pi)$$

$$- \frac{Gae}{2}.\sin.(5n't - 2nt + 5\epsilon' - 2\epsilon - \pi).$$

The two last terms may be made to have the same arguments as the preceding terms have, by making

$$Ha \cos. \pi - Ga \sin. \pi = E,$$
$$Ha \sin. \pi + Ga \cos. \pi = F,$$

in which case, they are to be thus expressed,

$$\frac{Ee}{2} \cos. (5nt - 2nt + 5\epsilon' - 2\epsilon)$$
$$+ \frac{Fe}{2} \sin. (5n't - 2nt + 5\epsilon' - 2\epsilon),$$

or we may still farther vary the expression, by making $\tan. A = \frac{F}{E}$, in which case the sum of the two last-mentioned terms equals

$$\sqrt{(F^2 + E^2)} . \frac{e}{2} \cos. (5n't - 2nt + 5\epsilon' - 2\epsilon - A).$$

In the preceding deduction of the value of δv, we have supposed the quantities P and Q to be constant. Now, P and Q are functions of the eccentricities, of the longitudes of the perihelia and nodes, and, also, (the planes of the orbit being supposed to be inclined to each other) of the inclination. These *elements* of the orbits, as they are called, are all subject to variations either secular or periodical; which, during 900 years, the period of that inequality which is the main subject of research in the present Chapter, may become considerable. It may be necessary then, to take account of them; and this, whether they be considerable or not, may be done by the following process.

Let R be restricted to denote those terms of its development which involve e^3, $e^2 e'$, $e' e^2$, e'^3, and which besides depend on the angle or argument $5n't - 2nt + 5\epsilon' - 2\epsilon$, then (see p. 333.)

$$R = m'P . \sin. (5n't - 2nt + 5\epsilon' - 2\epsilon)$$
$$+ m'Q . \cos. (5n't - 2nt + 5\epsilon' - 2\epsilon),$$

$$\text{and, } dR = -2n.dt \begin{Bmatrix} m'P \cos. (5n't - 2nt + 5\epsilon' - 2\epsilon) \\ -m'Q . \sin. (5n't - 2nt + 5\epsilon' - 2\epsilon) \end{Bmatrix}$$

$$\text{Now,} \quad \int Q.dt. \sin. (5n't - 2nt + 5\epsilon' - 2\epsilon) =$$

$$- \frac{Q}{5n' - 2n} \cos. (5n't - 2nt + 5\epsilon' - 2\epsilon) +$$

$$\frac{1}{(5\,n'-2\,n)^2}\int\frac{d\,Q}{d\,t}\,d\,t.\cos.\,(5\,n'\,t-2\,n\,t+5\epsilon'-2\epsilon),$$

and $$\int P\,d\,t.\cos.\,(5\,n'\,t-2\,n\,t+5\,\epsilon'-2\,\epsilon)=$$

$$\frac{P}{5\,n'-2\,n}\sin.\,(5\,n'\,t-2\,n\,t+5\,\epsilon'-2\,\epsilon)+$$

$$\frac{1}{(5\,n'-2n)^2}\int\frac{d\,P}{d\,t}\,d\,t\,.\sin.\,(5\,n'\,t-2\,n\,t+5\,\epsilon'-2\,\epsilon).$$

Now the last term of this second equation is given, in its *form* at least, by the first equation: and the last term of the first equation is similarly given by the second equation: so that, continuing the process,

$$\int Q\,d\,t\,.\sin.\,(5\,n'\,t-2\,n\,t+5\,\epsilon'-2\,\epsilon)=$$

$$-\frac{Q}{5\,n'-2\,n}\cos.\,(5\,n'\,t-2\,n\,t+5\,\epsilon'-2\,\epsilon)-$$

$$\frac{1}{(5\,n'-2\,n)^2}\,.\,\frac{d\,Q}{d\,t}\sin.\,(5\,n'\,t-2\,n\,t+5\,\epsilon'-2\,\epsilon)+\&c.$$

$$\int P\,d\,t.\cos.\,(5\,n'\,t-2\,n\,t+5\,\epsilon'-2\,\epsilon)=$$

$$\frac{P}{5\,n'-2\,n}\sin.\,(5\,n'\,t-2\,n\,t+5\,\epsilon'-2\,\epsilon)+$$

$$\frac{1}{(5\,n'-2\,n)^2}\,.\,\frac{d\,P}{d\,t}\cos.\,(5\,n'\,t-2\,n\,t+5\,\epsilon'-2\,\epsilon)-\&c.$$

and the same forms will serve for finding the second integrals, and, accordingly,

$$\int d\,t\int Q\,d\,t\,\sin.\,(5\,n'\,t-2\,n\,t+5\,\epsilon'-2\,\epsilon)=$$

$$-\frac{Q}{(5\,n'-2\,n)^2}\sin.\,(5\,n'\,t-2\,n\,t+5\,\epsilon'-2\,\epsilon)-$$

$$\frac{1}{(5\,n'-2\,n)^3}\,.\,\frac{d\,Q}{d\,t}\cos.\,(5\,n'\,t-2\,n\,t+5\,\epsilon'-2\epsilon)$$

$$+\ \&c.$$

$$\int d\,t\int P\,d\,t.\cos.(5\,n'\,t-2\,n\,t+5\,\epsilon'-2\,\epsilon)=$$

$$-\frac{P}{(5\,n'-2\,n)^2}\cos.\,(5\,n'\,t-2\,nt+5\,\epsilon'-2\,\epsilon)-$$

$$\frac{1}{(5\,n'-2\,n)^3}.\frac{d\,P}{d\,t}\sin.\,(5\,n'\,t-2\,n\,t+5\,\epsilon'-2\,\epsilon)$$

$$+\ \&c.$$

Hence, if the process be stopped short of the terms that involve $\frac{d^2P}{dt^2}$, $\frac{d^2Q}{dt^2}$, &c. there will result

$$3a\int n\,dt\int dR =$$

$$\frac{6am'n^2}{(5n'-2n)^2}\left\{\begin{array}{l}\left[P-\frac{2}{5n'-2n}\left(\frac{dQ}{dt}\right)\right]\cos.(5n't-2nt+5\epsilon'-2\epsilon)\\ -\left[Q+\frac{2}{5n'-2n}\left(\frac{dP}{dt}\right)\right]\sin.(5n't-2nt+5\epsilon'-2\epsilon)\end{array}\right\}$$

The quantities P and Q are easily determined. The general term of the development of R which involves the cubes, and the products of the eccentricities of three dimensions, when expanded, equals

$$\begin{array}{l} m'Le^3.\cos.(5n't-2nt+5\epsilon'+2\epsilon-3\pi)\\ +m'L'e^2e'.\cos.(5n't-2nt+5\epsilon'-2\epsilon-2\pi-\pi')\\ +m'L''ee'^2\cos.(5n't-2nt+5\epsilon'-2\epsilon-\pi-2\pi')\\ +m'L'''e'^3\cos.(5n't-2nt+5\epsilon'-2\epsilon-3\pi'); \end{array}$$

considering the planes of the two orbits as coincident: otherwise, P and Q would be functions also of the inclination.

If we now so reduce, according to the formulæ of Trigonometry, the above terms, that they involve solely the sines and cosines of the angle $5n't-2nt+5\epsilon'-2\epsilon$, and compare the reduced expression with the expression for R (see p. 333.) there will result

$$\begin{array}{l} P = Le^3.\sin.3\pi + L'e^2e'.\sin.(2\pi+\pi')\\ \quad + L''.ee'^2\sin.(\pi+2\pi') + L'''e'^3.\sin.3\pi',\\ Q = Le^3.\cos.3\pi + L'e^2e'.\cos.(2\pi+\pi')\\ \quad + L''ee'^2.\cos.(\pi+2\pi') + L'''e'^3.\cos.3\pi'. \end{array}$$

But we do not yet possess the means of computing all the terms of the preceding formula; for, $\frac{dP}{dt}$, $\frac{dQ}{dt}$, are not determined.

Since, however, P and Q are functions of e, π, θ, &c.; $\frac{dP}{dt}$, $\frac{dQ}{dt}$ will be so also. In order to find their value, compute P and Q for a particular æra; 1750 for instance: next, compute from the formulæ * by which the variations of e, π, &c. are expressed, the values (P', Q') of P and Q for another æra 1950: then, since

$$P' = P + t.\frac{dP}{dt} + \frac{t^2}{2}.\frac{d^2P}{dt^2} + \text{\&c.}$$

$$= P + t.\frac{dP}{dt}, \text{ nearly,}$$

$$\frac{dP}{dt} = \frac{P' - P}{t} = \frac{P' - P}{200}.$$

Similarly, $\dfrac{dQ}{dt} = \dfrac{Q' - Q}{200}$.

In the theory of Jupiter and Saturn, $5n' - 2n$, as it has been already stated, is a small quantity; about $\frac{1}{74}$th part of n. The terms, therefore, in δv that are divided by the square of $5n' - 2n$ must be much greater than the other terms: so much so, that they alone will serve to represent, with tolerable exactness, the inequality which is the present object of research; and, if so, then

$$\delta v =$$

$$\frac{6am'n^2}{(5n'-2n)^2}\left\{\begin{array}{l}\left(P - \dfrac{2}{5n'-2n}.\dfrac{dQ}{dt}\right)\cos.(5n't - 2nt + 5\epsilon' - 2\epsilon)\\ -\left(Q + \dfrac{2}{5n'-2n}.\dfrac{dP}{dt}\right)\sin.(5n't - 2nt + 5\epsilon' - 2\epsilon)\end{array}\right\}$$

Since n', m', in the preceding investigation, represent, respectively, the mean motion and mass of Saturn, the value of δv, just given, must represent an inequality affecting Jupiter's motion, and, since the period depends on $5n' - 2n$, of an extremely long period. But, as we may infer from p. 328. Saturn has a corresponding inequality, of an equal period, but different in its *affection* and degree. Whilst Saturn is retarded, Jupiter is accelerated. Now, it is plain, the inequality in Saturn's motion may be found exactly as that in Jupiter's has been, by a direct investigation of an integral similar to $3\,a\int n\,dt\int dR$; which integral is $3\,a'\int n'\,dt\int dR'$.

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* The general formulæ for determining the variations of the elements will be given in a subsequent Chapter.

We must now then find the value of R', or, rather, those terms of its value which involve the cubes of the eccentricities, and which are dependent on the angle $5\,n't - 2\,n\,t$.

Now, (see pp. 68, 273.), neglecting the inclination,

$$R' = \frac{m\,(x'x + y'y)}{r^3} - \frac{m}{\sqrt{(x - x')^2 + (y - y')^2}}$$

$$= \frac{m\,r'}{r^2}\cos.\,(v - v') - \frac{m}{\sqrt{[r'^2 - 2rr'\cos.\,(v - v') + r^2]}}.$$

Now we may at once reject the first term $\frac{m\,r'}{r^2}\cos.\,(v - v')$: for, on examining the values of r', $\frac{1}{r^2}$, and $\cos.\,(v - v')$ (see pp. 274, 278, &c.) it will be found to involve no terms such as we are in quest of, namely, terms involving e^3, $e^2 e'$, &c. and dependent on the angle $5\,n'\,t - 2\,n\,t$. The same is true (as, indeed, it has been proved by the very process of p. 278.) of the first term in R. The terms, therefore, that are the objects of enquiry, are to be sought for in the last terms of R and R', that is, in

$$\frac{m'}{\sqrt{[r'^2 - 2rr'\cos.\,(v' - v) + r^2]}}, \text{ and } \frac{m}{\sqrt{[r^2 - 2rr'\cos.\,(v - v') + r'^2]}},$$

but the denominators of these two fractions are the same: the terms, therefore, of the developments of the denominators will be the same: consequently, if

$$m'\,P\sin.\,(5\,n'\,t + 2\,n\,t + 5\,\epsilon' - 2\,\epsilon)$$
$$+\,m'\,Q\cos.\,(5\,n'\,t + 2\,n\,t + 5\,\epsilon' - 2\,\epsilon),$$

represent the terms in R which involve the cube of the eccentricity, &c. and which depend on the angle $5\,n'\,t - 2\,n\,t$,

$$m\,P\sin.\,(5\,n'\,t - 2\,n\,t + 5\,\epsilon' - 2\,\epsilon)$$
$$+\,m\,Q\cos.\,(5\,n'\,t - 2\,n\,t + 5\,\epsilon' - 2\,\epsilon),$$

will represent the like terms in R'. Hence, (since now it is $n't$ that varies)

$$d R' = 5 n' m P \cos.(5 n' t - 2 n t + 5 \epsilon' - 2 \epsilon)$$
$$- 5 n' m Q \sin.(5 n' t - 2 n t + 5 \epsilon' - 2 \epsilon),$$

and

$$3 a' \int\int n' d t . d R' =$$
$$- \frac{5 m n'^2 a'}{(5 n' - 2 n)^2} \left\{ \begin{array}{l} \left(P - \frac{2}{5 n' - 2 n} . \frac{dQ}{dt}\right) \cos.(5n't - 2nt - 5\epsilon' + 2\epsilon) \\ - \left(Q + \frac{2}{5 n' - 2n} . \frac{dP}{dt}\right) \sin.(5n't - 2nt + 5\epsilon' - 2\epsilon) \end{array} \right\}$$

This term, therefore, with the preceding, expounds, very nearly, the *great Inequality*, as it is called, of Jupiter and Saturn. If the latter expound Saturn's *retardation*, the former, affected with a different sign, will expound Jupiter's *acceleration*: and the contrary: the period of each is the same, exceeding 900 years; and, during half of this period, whilst the one inequality retards, by minute and almost insensible degrees, Saturn's motion, the other by like, though not equal, degrees accelerates Jupiter; this half period being elapsed, Saturn, during the remainder of the period, is accelerated, and Jupiter retarded. And, the effects of the modifications of the disturbing forces producing the above inequalities very nearly resemble those produced by uniformly accelerating and retarding forces: which is the reason why they were expounded by *secular* equations such as $A t^2$ *.

We have expounded δv, and $\delta v'$ by their principal terms ($3 a \int\int n d t . d R$, and $3 a' \int\int n' d t . dR'$), which are so from their involving the divisor $(5 n' - 2 n)^2$; the inequality, therefore, of Jupiter, or, *Jupiter's acceleration*, is to the corresponding inequality of Saturn, or, *Saturn's retardation*, as $6 m' n^2 a$ is to

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* If, without finding them to be periodical, it had been found that the inequalities were not produced by uniform acceleration and retardation, their effects would probably have been expounded by empirical equations, such as $A t^2 + B t^3$, or $A t^2 + B t^3 + C t^4$.

$-15\,m\,n'^2\,a'$: which quantities are to each other very nearly as 3 to 7; a result the same as that which in p. 329. was otherwise deduced, and which is confirmed by observation.

The theory of gravity, then, explains, as far at least as regards their general nature and character, the *great inequalities* of Jupiter and Saturn: and, accordingly, the corresponding *accelerations* and *retardations* of these two planets are no longer anomalous phenomena. They arise from the mutual attraction of the two planets, are very minute in degree, and become sensible only by accumulation of effect during a very long period.

But a more severe proof is required of the solution of phenomena on Newton's system, than the explanation of their general nature and character: and, in the present case, it ought to be shewn that, on taking account of the inequality expounded by

$$-\frac{15\,m\,n'^2\,a'}{(5n'-2n)^2}\left\{\begin{array}{l}\left(P-\dfrac{2}{5n'-2n}\cdot\dfrac{dQ}{dt}\right)\cos.(5n't-2nt+5\epsilon'-2\epsilon)\\ -\left(Q+\dfrac{2}{5n'-2n}\cdot\dfrac{dP}{dt}\right)\sin.(5n't-2nt+5\epsilon'-2\epsilon)\end{array}\right\}$$

Saturn's mean motion ought to result the same in value from the comparison of modern observations, as from the comparison of observations 2000 years distant.

Observations so distant as the last must give the mean motion very exactly. The two oppositions made use of for determining Saturn's motion, were, one recorded to have happened 228 years A. C., the other observed in 1715. The interval is 1943 years. In that interval the greatest possible effect of Saturn's inequality could not exceed 1° 33′ 40″, and, consequently, the error in estimating the mean annual motion could not exceed 3″.

If we find the numerical values of P, Q, $\frac{dP}{dt}$, $\frac{dQ}{dt}$, then the expression for the mean longitude of Saturn, corrected solely on account of the inequality which has been made the subject of discussion, is

$$n't+\epsilon'-\frac{15\,n'^2\,m}{(5n'-2n)^2}\begin{Bmatrix} .00111965\ \text{sin.}\,(5\,n't-2\,nt+5\,\epsilon'-2\,\epsilon) \\ +.00010917\ \text{cos.}\,(5\,n't-2\,nt+5\,\epsilon'-2\,\epsilon) \end{Bmatrix}$$

or, if we proceed still farther, substitute for n, n' their values, and reduce the two terms to one, Saturn's mean longitude, reckoned from the equinox of 1750, will be

$$n'\,t+\epsilon'-49'\,13''\ \text{sin.}\,(5\,n't-2\,nt+5\,\epsilon'-2\,\epsilon+5^{\circ}\,34'\,8'')\,*.$$

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* *First*, $\frac{15\,n'^2\,m}{(5\,n'-2n)^2}$ *computed.*

$n'=43996\ldots\ldots\ 2\ \text{log.}=9.2868264$

$\text{log.}\ 15=1.1760913$

$10.4629177\ (a)$

$5\,n'-2\,n=1462''\ldots\ldots\ 2\ \text{log.}=6.3298948$

$\frac{1}{m}=1067.195\ldots\ldots\ \text{log.}=3.0282458$

$9.3581406\ (b)$;

$\therefore\ (a)-(b)=1.1047771$ is the log. of $\frac{15\,n'^2\,m}{(5n'-2n)^2}\ (=F)$,

next, in order to reduce the two terms to one, let them be represented by

$$F.(A\ \text{sin.}\ Nt+B.\text{cos.}\ Nt)$$

$$=FA.\left(\text{sin.}\ Nt+\frac{B}{A}\ \text{cos.}\ Nt\right)$$

$$=\frac{FA}{\text{cos.}\ \theta}\ \text{sin.}\,(Nt+\theta),\quad \text{making tan.}\ \theta=\frac{B}{A}.$$

Tan. θ computed.

$\text{log.}\ r=\text{log.}\ 10\ldots\ldots\ldots\ldots\ldots\ldots\ 10$

$\text{log.}\ B=\text{log.}\ .00010917\ldots\ldots=\bar{4}.0381033$

$\text{log.}\ A=\text{log.}\ .00111965\ldots\ldots=\bar{3}.0490824$

$8.9890209=\text{log. tan.}\ (5^{\circ}\ 34'\ 8'')$;

$\therefore\ \theta=5^{\circ}\ 34'\ 8''.$

Lastly,

Suppose this formula to be exact, then Saturn's mean longitude for the year 1715 is to be derived from it by making $t = -35$, in which case, it becomes

$$-35\,n' + \epsilon' - 49'\ 13''\ \text{sin.}\ (60^{\circ}\ 15'\ 28''):$$

and his longitude for the year 1595, making $t = -135$, will become

$$-135\,n' + \epsilon' - 49'\ 13''.\ \text{sin.}\ (11^{\circ}\ 19'\ 28'')\ *,$$

and if, according to the method of finding the mean motion (see *Astron.* p. 262.) we divide the difference of these two longitudes by the interval elapsed (120 years), the mean motion will result,

$$n' - \frac{49'\ 13'' \times .6716}{120},$$

.6716 being, nearly, the difference of the natural sines of the arcs 60° 15′ 28″, and 11° 19′ 28″.

$$\textit{Lastly,}\ \frac{\text{F}\,.\,\text{A}}{\text{cos.}\ \theta}\ \textit{computed.}$$

$$\begin{aligned}
\text{log.}\ F &= 1.1047771\\
\text{log.}\ A &= \bar{3}.0490824\\
\text{log.}\ (\text{arc}=\text{rad.}) &= 5.3144251\\
\hline
&\ 3.4682846\\
\text{log. cos.}\ 5^{\circ}\ 34'\ 8'' &= 9.9979470\\
\hline
&\ 3.4703376 = \text{log.}\ 2953'' = \text{log.}\ 49'\ 13'',\ \text{nearly.}
\end{aligned}$$

* The computation is thus effected: for the epoch of 1750,

$$\begin{aligned}
\epsilon' &= 7^{s}\ 21^{\circ}\ 17'\ 20'',\\
\epsilon &= \phantom{7^{s}\ 2}3\ \ 44\ \ 30;\\
\therefore\ 5\epsilon' - 2\epsilon &= 38\,.\,8\ \ 57\ \ 40,
\end{aligned}$$

$$\text{for 100 years}\ (5\,n' - 2\,n')\ 100 = 40^{\circ}\ 46'\ 40'';$$

$$\begin{aligned}
\therefore\ \text{for}\ 35 \ldots\ldots\ldots\ldots\ldots &= 14\ \ 16\ \ 20\\
\text{for}\ 135 \ldots\ldots\ldots\ldots\ldots &= 63\ \ 13\ \ 20,
\end{aligned}$$

consequently, for the year 1595,

$$\begin{aligned}
5\,n't - 2\,nt + 5\,\epsilon' - 2\,\epsilon + 5^{\circ}\ 34'\ 8'' \ldots\ldots\ldots\ldots\ldots &= 36^{s}\ 11^{\circ}\ 19'\ 28''\\
\text{and for the year 1715} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots &= 36\ \ 60\ \ 15\ \ 28.
\end{aligned}$$

The value of $49' \; 13'' \times . \frac{6716}{120}$ is nearly $16''.5$, consequently, Saturn's mean annual motion, determined by the above observations, ought, according to theory, to appear *retarded* by about sixteen seconds: which, according to Lalande, agrees with observation: that is, the oppositions of 1595 and 1705, assign to Saturn a mean annual motion less by about sixteen seconds, than the oppositions of 228 A. C. and 1715 (see p. 325.).

In the preceding process (see p. 344.) by which the formula for $\delta v'$ was reduced to one term, the values of P, Q, &c. were computed for 1750: for a different epoch they would have different values: since they are functions of e, π, &c. which are variable. If then with altered values (P', Q', &c.) of P and Q, a reduction of the two terms into one, similar to that of p. 344. were made, the coefficient of the resulting formula, and the arc added to $5n't - 2nt + 5\epsilon' - 2\epsilon$ would be different: for the year 228 A. C., for instance, the coefficient would be about $-52'$, and the arc about 40^0: for the year 1950, the coefficient would be $-48' \; 52''$, and the arc $2^0 \; 17' \; 52''$: the coefficient and arc would both decrease: and if they were supposed to decrease uniformly $-(59' \; 13'' - p.s'')$ might represent the first, and

$$5^0 \; 34' \; 8'' - p.A^0,$$

the second: p being the number of years to be reckoned from 1750, and s'', and A^0 being, respectively, the number of seconds, and the arc, by which the coefficient and the original arc ($5^0 \; 34' \; 8''$) are diminished in one year. And, in this case, s'' and A^0 might easily be determined: since, for that purpose, we have, from the epochs of 1750 and 1950,

$$49' \; 13'' - 200.s'' = 48' \; 52'',$$
$$5^0 \; 24' \; 8'' - 200.A^0 = 2^0 \; 17' \; 52'';$$

$$\text{whence, } s'' = 0''.105, \text{ and } A^0 = 58''.88:$$

so that the formula for correcting Saturn's mean longitude would be

$$-(49' \, 13'' - p \times 0''.105) \sin. (5n't - 2nt + 5\epsilon' - 2\epsilon + 5^0 \, 34' \; 8'' - p.58'.8),$$

the epoch being 1750, and p being reckoned negative or positive, accordingly as the year, on which the value of the above correction should be required, should be under or above 1750.

In the volume of the Memoirs of the *Academy of Paris* for 1785 (which volume contains Laplace's original researches on this subject) the formula for the correction of Saturn's longitude is

$$\left.\begin{matrix} {}^{*}\,-\,48'\,44'' \\ -\,p \times 0''.10836 \end{matrix}\right\} \sin.\,(5n't - 2nt + 5\epsilon' - 2\epsilon + 5^{\circ}\,34'\,8''),$$

which M. Laplace says may be extended to 2000 years before, and about 1200 years after 1750.

If we multiply the coefficient of the preceding term by $-\frac{3}{7}$, we shall obtain the coefficient of the term expounding Jupiter's *acceleration*, which has the same argument, and, accordingly, the same period as Saturn's *retardation*. *Acceleration* and *retardation*, being, in this subject, as it has been more than once explained, the technical denominations of effects expounded by certain of the terms that involve the cubes of the eccentricities, or, generally, those products of the eccentricities and inclination that are of three dimensions.

The period of this *great inequality* is of some interest. Its great length may be considered as the efficient cause of the embarrassment of Astronomers when they were noting the anomalous phenomenon of the retardation of Saturn's mean motion.

Its length may be determined on the principles of p. 235. Since the argument is

$$5n't - 2nt + 5\epsilon' - 2\epsilon - 5^{\circ}\,34'\,8'' - p \times 58''.88,$$

in which the variable part is

$$5n't - 2nt - p \times 58''.88,$$

we must investigate that value of t (or p) which will make the above variable part = 360°. Make t and $p = 1$, then

$$5n' - 2n - 58''.88 = 1410''.6,$$

---

* 48′. 44″ is Delambre's coefficient.

consequently,

$$t = \frac{360^{0}}{1410''.6} = 918\tfrac{3}{4}, \text{ nearly.}$$

It can, therefore, be no matter of surprise, that Astronomers, by the comparison of mere observations alone, should have been unable to disentangle from phenomena (the phenomena of observed longitudes) an inequality, which, in the beginning of its agency, would not affect the longitude by more than one or two seconds, and the accumulated effect of which, during 450 years, would never reach 50 minutes.

The retardation of Saturn's mean motion may be supposed, as it has been said, to proceed from a modification of the disturbing force, resembling, in its effects, an uniformly retarding force. In such a supposition then, a term, like At^2 which expounds the effect of a force, either uniformly retarding or accelerating, would expound that of the disturbing force. In astronomical language it would expound a *secular equation* to be used for the purpose of correcting Saturn's mean motion. But, the disturbing force (that modification of it which is the cause of the great inequality that has been treated of) does not strictly resemble an uniformly retarding force: its intensity, always very small, yet varies from year to year: and, accordingly,

$$A t^2 + B t^3 + C t^4 + \&c.$$

would more truly represent its effect than a single term such as $A t^2$.

The expression for Saturn's retardation is (see p. 346.)

$$\left.\begin{matrix} (49' \;\; 13'' \\ - p \times 0''.105) \end{matrix}\right\} \sin.(5n't - 2nt + 5\epsilon' - 2\epsilon + 5^{0}\,34'\,8'' - p \times 58''\,58),$$

and it is easy to deduce from it its variation, and thence to determine when the variation is a maximum: for example, its variation (expressed by its differential coefficient),

$$\left.\begin{matrix} (49' \;\; 13'' \\ - p \times 0''.105)\;23' \end{matrix}\right\} \cos.(5n't - 2nt + 5\epsilon' - 2\epsilon + 5^{0}\,34'\,8'' - p . 58''.58),$$

which quantity would be at its maximum when

$$5n't - 2nt + 5\epsilon' - 2\epsilon + 5^0\ 34'\ 8'' - t \times 58''.58 = 0,$$

and the value of t resulting from this equation is nearly 190, which, subtracted from 1750, leaves 1560 to represent the epoch at which Saturn's retardation, during the year, was the greatest, and equal to

$$\begin{aligned} &(49'\ 13'' - 2''.09)\ 23'\ 30'' \\ &= (49'\ 13' - 2''.09)\ \frac{\text{sin. } 23'\ 30''}{\text{radius}} \\ &= 20'', \text{ nearly.} \end{aligned}$$

About the year 1560, then, the observations must necessarily have shewn Saturn, in his greatest retardation, and (see p. 347.) Jupiter in his greatest acceleration. Or, with reference solely to their *great inequality*, the true motions of the above two planets differed most from their mean about the time of Tycho Brahe's observations; an epoch remarkable for the revival of Astronomical Science.

The explanation of Saturn's retardation, according to the law and principle of Gravity, was first given by Laplace in the *Memoirs of the Academy of Paris* for the years 1785, 1786. His researches, however, go beyond that explanation; and are extended to comprehend the complete theory of Jupiter and Saturn, or, what is in fact, a general solution of the *Problem of the Three Bodies*.

The solution of the *great inequality* of Jupiter and Saturn, on the principles of Physical Astronomy, marks, with considerable precision, the progress which that science has made since its rise. Its great founder, as we have remarked, (see p. 326.) noted no peculiarity in the theory of Jupiter and Saturn: nothing, which either strongly confirmed, or which seemed to form an exception to, his system. Yet, as we have seen, there are, in the motions of the above two planets, remarkable inequalities: which, for a long time were considered as anomalous, and which, as long as they were so considered, formed an exception amongst the results from the law of Gravity. They, however, most forcibly illustrated the truth of that law when they were proved not to be anomalous, and had been reduced to the class of other inequalities that arose from planetary perturbation.

The minute and very gradual variations and long periods of those inequalities that are the subject of the present discussion, have occasioned the practical difficulty of detecting them by observation. The difficulty of detecting their mathematical cause, has arisen from its lying concealed, as it were, amongst insignificant terms.

The cause, which in the theory of Jupiter and Saturn, gives importance to certain of these terms may operate in other cases. If the mean motions of any two planets are nearly commensurable, such planets are subject to inequalities of a very long period. But there is no rule, short of actual trial, for ascertaining whether any two planets are under the predicaments that Jupiter and Saturn are. We must examine the Table of Mean Motions, and, on making trials, we shall find (see *Astronomy*, p. 283.) that five times Mercury's period is nearly equal to twice Venus's, or, which is the same thing, twice the mean motion of Mercury (n) is nearly equal to five times that of Venus (n');

$$\therefore\ 2n - 5n' = 0, \text{ nearly, or}$$

$$3n - 5n' = n, \text{ nearly.}$$

Now, some of the terms, in that part of the expression for $\frac{r\,\delta r}{a^2}$ (see p. 333.) which involves the squares of the eccentricities, depend on the angle $3nt - 5n't$: and (see p. 333.) the integration introduces into the value of $\frac{r\,\delta r}{a^2}$, the divisor,

$$(3n - 5n')^2 - n^2 = (2n - 5n')(4n - 5n'),$$

which by reason of the factor $2n - 5n'$, becomes very small: and accordingly, in the theory of Mercury disturbed by Venus, it is necessary to attend to the inequality dependent on the angle $3nt - 5n't$.

In the same manner, since the mean motion of Mercury is nearly equal (see *Astronomy*, p. 283.) to four times that of the Earth, we must attend, in the theory of Mercury disturbed by the Earth, to the inequality dependent on the angle $2nt-4n''t$ in the terms of the expression for $\frac{r\delta r}{a^2}$, that involve the squares of the eccentricities: for, here, the divisor introduced by integration is $(2n - 4n'')^2 - n^2$, which contains the factors $n - 4n''$, and $3n - 4n''$, the first of which is very small.

The terms just spoken of, involve the squares of the eccentricities and form part of the value of $\frac{r\delta r}{a^2}$: but, if $n-4n''$ be very small, it will be necessary (see pp. 335, &c.) to attend to the terms in the expression for δv which involve the cubes of the eccentricities, and depend on the angle $nt-4n''t$: for, in such terms, a divisor $(n - 4n'')^2$ is admitted.

The explanation of the alternate retardation and acceleration of Saturn and Jupiter affords, it has been said, a kind of practical proof of the progress which *Physical Astronomy* has made since the time of Newton: it shews also after what manner the latter science is superior to *Plane Astronomy*, and is capable of benefiting it. For although we may, by the aid of empirical equations and observations, determine inequalities of short periods, yet it seems impossible, by like means, to determine inequalities so protracted in their periods as those we have been discussing.

Observation is unequal to the task of disengaging them: they would always, without the aid of theory, appear so blended with the mean motions as either gradually to accelerate or retard them. Modern observations, for instance, compared with each other, and with antient observations, would, as Halley found it, make Saturn to appear retarded and Jupiter accelerated: or, modern observations, compared with each other, might, as Lambert found it to be the case, make Jupiter appear retarded and Saturn accelerated. The Tables of these two planets, before the causes, laws and quantities of their *accelerations* and *retardations* were ascertained, were erroneous to the amount of twenty-two minutes. Laplace's equations reduced the errors within two minutes; and the Tables are now exact to within a quarter of a minute: this is one of the practically good effects of theory: and in this, as in similar instances, it no longer goes hand in hand with observation, but advances before and serves it as a guide *.

By the theorem of p. 328. it follows, that if Saturn were subject to a *really secular* retardation from the action of Jupiter, Jupiter would suffer an acceleration equally secular from Saturn, and in the proportion of $m' \sqrt{a'} : -m\sqrt{a}$. Now the inequalities, as we have seen in p. 348. are not secular but periodical: their periods, however, are so long, that the inequalities are almost accurately in the above proportion: they would be less accurately so, were their periods shorter: still, however, as the fact is, not very inaccurate, were the periods very much shortened.

Hence, if we should have investigated the *acceleration* produced, during a considerable period, in one planet's motion by the action of another, we might at once find the corresponding retardation produced, during the same period, in the motion of the disturbing body, by merely multiplying the first found acceleration by $-\frac{m\sqrt{a}}{m'\sqrt{a'}}$: or, this last result might be used as a test of the truth of the retardation computed by a direct process. For instance, the action of the Earth on Venus causes an inequality dependent on the angle $3n''t - 2n't$: (n'', n' denoting the mean motions of the Earth and Venus): the period of which, accordingly, (see p. 235, and *Astron.* p. 283.) is nearly four years, the inequality, expressed by its two parts †, is

$$1''.5\,.\sin.(3n''t - 2n't + 3\epsilon'' - 2\epsilon' - \pi')$$
$$-4''.5\,.\sin.(3n''t - 2n't + 3\epsilon'' - 2\epsilon' - \pi''):$$

this, multiplied by $-\frac{m'\sqrt{a'}}{m''\sqrt{a''}}$, gives

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* There is something curious in the history of the theory of Jupiter and Saturn. First, its peculiar phenomena were unnoticed by the great founder of Physical Astronomy: next, when noted and examined they seemed to impair his system; lastly, they have served, when explained and accounted for, most strongly to confirm it.

† This inequality might be expressed by a single term by means of the process of p. 344.

$$-1''.03 . \sin . (3 n'' t - 2 n' t + 3 \epsilon'' - 2 \epsilon' - \pi')$$
$$+ 3''.48 . \sin . (3 n'' t - 2 n' t + 3 \epsilon'' - 2 \epsilon' - \pi''),$$

for the corresponding inequality in the Earth's motion caused by the action of Venus: which is, very nearly, an accurate result, since the two coefficients, deduced by a direct process, are $-1''.08$, $+3''.6$.

In like manner the action of the Earth on Venus produces an inequality dependent on the angle $5 n'' t - 3 n' t$, and the period of which, accordingly, is about eight years. The inequality expressed by one term, is

$$-1''.5 . \sin . (5 n'' t - 3 n' t + 5 \epsilon'' - 3 \epsilon' + 20^{\circ}\ 54'\ 28''),$$

and, if this denote a *retardation*, the coefficient of the corresponding *acceleration* in the Earth's motion produced by the action of Venus, is

$$-1''.5 \times -\frac{m' \sqrt{a'}}{m'' \sqrt{a''}} = 1''.1,$$

which is very nearly the value of the coefficient resulting from the direct process.

If Venus then accelerate the Earth by a particular inequality for a certain period, the Earth will retard Venus by a like inequality and for the same period: the coefficient of the inequality will be different, the argument the same. This we know certainly by the mathematical process. But the thing admits somewhat of a popular explanation, if we suppose these inequalities to originate from modifications of the tangential disturbing forces: for then, if Venus should tend to draw the Earth forward in its orbit, the Earth must tend to draw Venus back: at the same time and for equal times, but not by equal degrees: since the accelerating force of Venus on the Earth, would, from her smaller mass, be less than the Earth's retarding force of Venus; but the correspondent *accelerations* and *retardations* of the mean motions will not, for obvious reasons, be necessarily proportional to the masses of the retarding and accelerating bodies. They follow, as we have seen in one case, see p. 38. a different ratio which must be ascertained by calculation. Indeed the preceding statement, as it was said in its outset, serves merely the purpose of popular explanation, and affords little else than a glimpse and indistinct view (*un apperçu*) of the subject.

The solution of the Problem of Three Bodies, it is sometimes stated in the sweeping clauses of indolent generalisers, comprehends every case of lunar and planetary disturbance. How delusive such a statement is, may be understood from the preceding pages. The methods of solutions used in the lunar theory will not apply, without considerable modifications, to the planetary: which modifications amount, in some instances, to the inventions of new methods. Again, the methods which apply to some of the planets will not apply to all: if we use the same formulæ, to the same extent, for Jupiter and Saturn, which are sufficient for Mars and Jupiter, we shall be sure of being wrong: or, rather, there will be produced results so anomalous as to make Newton's theory appear inadequate to the explanation of all the planetary phenomena. In fact, the *natural* complication, if we may so express ourselves, of the subject is such, that we cannot safely predict what cases are strictly similar. Each requires a separate examination, during which, new methods are continually suggesting themselves. Analysis has been furnished with some of its excellent formulæ from the differences found to exist between the lunar and planetary theories.

Although, therefore, we have gone through the lunar and planetary theories, we are not warranted, by the experience of what has preceded, in supposing that the methods there used will strictly apply to the system of Jupiter and his satellites, or to that of Saturn and his.

The drift of an enquiry into the perturbations of these satellites will be to find out what is peculiar to them: it is evident their theory possesses many points of similarity with the planetary theory. The system of Jupiter and his satellites, has, indeed, not inaptly, been said to be the Solar System in miniature. To every case in the latter, we may find an analogous one to the former: for instance, a satellite of Jupiter disturbed by the Sun's action is a case altogether analogous, except in being more simple, to that of the Moon disturbed by the action of the Sun. Again, one of Jupiter's satellites disturbing his orbit is a case analogous to that of the Moon disturbing the Solar Orbit, and which has been treated of in Chapters VI, and XVI. Thirdly, the *mutual* perturbations of the first and fourth satellites are analogous to the mutual perturbations of Venus and Jupiter, or of the Earth and Jupiter, and require merely, for their mathematical investigation, the simple processes of pp. 261, &c. But the mutual perturbations of the first and second, inasmuch as they require the peculiar computation described in Chapter XVIII. (for, ☾′, ☾″ representing the first and second satellite, $\frac{\text{rad. orbit of ☾}'}{\text{rad. orbit of ☾}''} = \frac{5.698491}{9.066548}$) are analogous to the perturbations of Venus and the Earth.

But is there any thing in the theory of Jupiter and his satellites analogous to that which has been noted as peculiar in the theory of Jupiter and Saturn? We mean those minute inequalities of a long period which arise from the near commensurability of the mean motions. Such inequalities in the theory of Jupiter are minute, since they depend on terms involving the squares and cubes of the eccentricities: that theory contains no other like inequalities either independent of the eccentricities, or involving their simple powers: since $2n$ being nearly $= 5n'$, the only terms that become large by integration, are those which admit the divisors $2n - 5n'$, and $(2n - 5n')^2$ (see p. 331.) which terms involve e^2, in the expression for $\frac{r\delta r}{a^2}$, and e^3 in the expression for δv (see pp. 334, 336.). But if n should nearly $= 2n'$, then the terms in the expression for $\frac{r\delta r}{a^2}$ which have, for their argument, $2n't - 2nt + 2\epsilon' - 2\epsilon$, and which (see p. 279.) are independent of the eccentricity, become large by receiving, from integration, the divisor $4(n' - n)^2 - n^2 = 4(2n' - n)(2n' - 3n)$, which is small inasmuch as $2n' - n$ is. Now this happens in the system of Jupiter's satellites: the mean motion of the first satellite is nearly double that of the second. So that, to go no farther, we have the instance of an inequality, in some sort, similar to that inequality of Jupiter and Saturn which has been the subject of the present Chapter. But the similarity is not exact: the distinguishing and peculiar circumstance in the theory of the first and second satellite is this, that those terms which receive divisors such as $n - 2n'$ are so large and *predominant* as alone to be adequate to represent the inequalities that arise from mutual perturbation. The other terms dependent on the angular distance of the two satellites may be neglected, as representing inequalities too small to be discerned by observation. This is not the case with Jupiter and Saturn: their *great inequality* (great from the length of its period) is much less than most of the inequalities which are either independent of the eccentricities, or which depend on their simple powers.

The mean motion of the second satellite is nearly double that of the third: there must arise, therefore, from their mutual perturbations, inequalities of that kind to which the first and second satellite are subject. The second satellite then must receive, both from the action of the first and third, an inequality of the same kind, and of that peculiar kind which has been already described: and, from the combination of the two inequalities, there arises a new inequality distinct from any that have hitherto been enumerated, and to which there is nothing analogous in the planetary theory.

The inequality just mentioned does not easily admit of a popular explanation. There are in Physical Astronomy, as in other branches of Science, many things so technical as to require a technical explanation. But were it otherwise, it would be a waste of time now to attempt to describe briefly, what it is purposed to explain with fulness in the succeeding Chapter.

CHAP. XX.

ON THE THEORY OF THE SATELLITES OF JUPITER.

Deduction of the Value of R: *First, when the Sun, secondly, when a Satellite, is the disturbing Body. Values of the Inequalities in Longitude and Parallax of a Satellite. Variation in a Satellite's Longitude arising from the Sun's disturbing Force. By reason of the near Commensurability of the Mean Motions of the Three first Satellites, their Inequalities in Longitude expressed, each, by a single Term. The Inequalities of the Second Satellite arising from the Actions of the First and Second Satellite blended together and expounded by a single Term. The Period of the Inequalities of the Three first Satellites* $= 437^d\ 15^h\ 48^m\ 57^s$. *The Elements of the Theory of the Satellites determined from the Epochs and Durations of their Eclipses.*

In the following investigations it is intended to use the differential equations of Chapter XVI.

The quantity R in those equations is used for the convenient expression of the disturbing force. By means of it, the Sun's disturbing force on any one of the satellites may be separately expressed: so may the disturbing force of one satellite on any other of the system: and, consequently, by the collection of similar values we may express the whole disturbing force acting on any one of the satellites.

To begin with the expression for the Sun's disturbing force on the first satellite,

Let

S be the Sun's mass,
D his distance,
U his longitude seen from the centre of Jupiter,
r the radius of the orbit of the first satellite,
v its longitude;

and, consequently, see pp. 66, 273,

$$R = \frac{S.r}{D^2}\cos.(U-v) - \frac{S}{\sqrt{[r^2 - 2rD\cos.(U-v) + D^2]}}.$$

Now it is unnecessary to expand this expression into a series, such as

$$\frac{1}{2}A + B\cos.(U-v) + C.\cos.2(U-v) + \&\text{c. (see p. 259.)}$$

Since from the smallness of $\frac{r}{D}$ $\left(\text{a quantity smaller than } \frac{\text{rad. ☽'s orbit}}{\text{rad. ⊕'s orbit}}\right)$ R may be, at the least, as simply expressed, as in the Lunar Theory: rejecting then (see p. 59.) the terms that involve $\frac{r^2}{D^2}$, $\frac{r^3}{D^3}$, &c., we have

$$R = -\frac{S}{D} - \frac{S\,r^2}{4\,D^3}[1 + 3\cos.2(U-v)].$$

If the second satellite be the disturbing body, and m', v', r' be its mass, longitude, and the radius of its orbit, the corresponding value of R, will be

$$R = \frac{m'r}{r'^2}.\cos.(v'-v) - \frac{m'}{\sqrt{[r^2 - 2rr'\cos.(v'-v) + r'^2]}},$$

which expression admits no such simple reduction as the preceding one does, since, now

$$\frac{r}{r'}\text{ being} = \frac{5.698491}{9.066548} = .6285183,\ \left(\frac{r}{r'}\right)^2,\ \left(\frac{r}{r'}\right)^3,\ \&\text{c.}$$

cannot be neglected.

Expressions similar to the last obtain for R, when the third and fourth satellite are the disturbing bodies; and, since the first satellite is really disturbed both by the Sun and by the other satellites, we must in estimating its perturbations, express R by the sum of its partial values, and then

$$R = -\frac{S}{D} - \frac{S \cdot r^2}{4 D^3}[1 + 3 \cos. 2 (U - v)]$$
$$+ \frac{m' r}{r'^2} \cos. (v' - v) - \frac{m'}{\sqrt{[r^2 - 2 r r' \cos. (v' - v) + r'^2]}}$$
$$+ \frac{m'' r}{r''^2} \cos. (v'' - v) - \frac{m''}{\sqrt{[r^2 - 2 r r'' \cos. (v'' - v) + r''^2]}}.$$
$$+ \&c.$$

and, for the purpose of computing $\int dR$, $\frac{r\, d R}{d r}$, the second, third, &c., lines of the preceding expression, must be expanded into series, such as

$$\frac{1}{2} A + B \cos. \omega + C . \cos. 2 \omega + \&c.$$

and A, B, C, &c. may be all computed by the methods given in Chapter XVIII.

From the preceding expression for R, all the disturbing forces that act on the first satellite may be computed: except indeed we consider it as subject to an additional perturbation arising from the *non*-sphericity of Jupiter.

The formulæ for R, by which the disturbing forces, acting either on the second, third, or fourth satellite, may be expressed, will be, it is plain, exactly similar to the preceding.

In order to find the values of $\frac{r \delta r}{a^2}$ and δv, it is necessary previously (see Chap. XVI.) to know those of $\int dR$, and $r \frac{d R}{d r}$; and, in order to determine these latter from the preceding expression for R, we must express it in terms of the mean distances and mean motions.

With regard to the first line in the above value of R, let

$$\frac{S}{D^3} = M,$$
$$U = M t + E,$$
$$v = n t + \epsilon,$$

then $\cos. 2(U - v) = \cos. 2(Mt - nt + E - \epsilon)$, and (see p. 264.)

$$2\int dR + r\frac{dR}{dr} =$$

$$-3M^2 a^2 . \frac{2n - M}{2n - 2M}\,[\cos. 2(nt - Mt + \epsilon - E)].$$

With regard to the second line in the value of R, and the other lines that would be similar to it, we must convert them into series (see pp. 273, 274.), such as

$$\frac{1}{2}\mathrm{A} + \mathrm{B}\cos.\omega + \Gamma\cos. 2\omega + \&c.$$

$$\frac{1}{2}\mathrm{A}' + \mathrm{B}'.\cos.\omega' + \&c.$$

ω, ω', &c. being $n't - nt + \epsilon' - \epsilon$, $n''t - nt + \epsilon'' - \epsilon$, &c. respectively, the orbits being supposed to be circular.

With this value of R, that of $2\int dR + r\frac{dR}{dr}$ will be exactly similar to the one deduced in p. 306, so that with this and the preceding one of the present page, the form of the differential equation will be as follows:

$$d^2 . \frac{r\delta r}{a^2 . dt^2} + n^2 . \frac{r\delta r}{a^2} + 2n^2 k - M^2$$

$$-3M^2 . \frac{2n - M}{2n - 2M}\cos.(2nt - 2Mt + 2\epsilon - 2E)$$

$$+ m'n^2 . \left\{ \begin{array}{l} \left(a^2 . \frac{d\mathrm{B}}{da} + \frac{2n}{n - n'} . a\mathrm{B}\right)\cos.(n't - nt + \epsilon' - \epsilon) \\ + \left(a^2 . \frac{d\Gamma}{da} + \frac{2n}{n - n'} . a\Gamma\right)\cos. 2(n't - nt + \epsilon' - \epsilon) \end{array} \right\}$$

$$+ \&c.$$

In taking the integral of this equation the coefficients of the terms dependent on the angles $n't - nt + \epsilon' - \epsilon$, $2n't - 2nt + 2\epsilon' - 2\epsilon$, must (see pp. 100, &c.) be respectively divided by $(n' - n)^2 - n^2$, $2(n' - n)^2 - n^2$ the coefficient of $\cos.(2nt - 2Mt + 2\epsilon - 2E)$ must be divided by $(2n - 2M)^2 - n^2$, or, from the relative minuteness of M, by $3\,n^2$: so that if, from the same cause of the minuteness of M*, we make $\frac{2\,n - M}{2\,n - 2\,M} = 1$, the coefficient will become $-\frac{M}{n^2}$.

The value of $\delta\,v$ from the expression of p. 268, and by the process of p. 307. will be

$$\delta\,v =$$

$$\left(3\,k - \frac{7}{4}\,\frac{M^2}{n^2} + \text{\&c.}\right) n\,t$$

$$+ \frac{11}{8}\,\frac{M^2}{n^2} \sin.\,(2\,n\,t - 2\,M\,t + 2\,\epsilon - 2\,E)$$

$$+ \frac{m'\,n}{n - n'} \times$$

$$\left\{\begin{array}{l} \left[\frac{n}{n-n'}\,.\,a\mathrm{B} + \frac{2\,n^2}{(n-n')^2 - n^2}\left(a^2\,.\,\frac{d\,\mathrm{B}}{d\,a} + \frac{2n}{n-n'}\,.\,a\,\mathrm{B}\right)\right] \sin.\,(n'\,t - n\,t + \epsilon' - \epsilon) \\ + \frac{1}{2}\left[\frac{n}{n-n'}\,a\Gamma + \frac{2\,n^2}{4\,(n-n')^2 - n^2}\left(a^2.\frac{d\Gamma}{d\,a} + \frac{2n}{n-n'}.\,a\,\Gamma\right)\right] \sin.\,2(n'\,t - n t + \epsilon' - \epsilon) \\ + \text{\&c.} \end{array}\right\}$$

The quantity k, as in p. 269. is introduced by integration. In order to determine it, make the coefficient of the second term equal 0, and there results,

$$k = \frac{7}{12}\,\frac{M^2}{n^2}, \text{ nearly.}$$

The term in the second line of the preceding expression for $\delta\,v$ expounds the inequality arising from the Sun's action. Now, as it is plain, the perturbation of a satellite of Jupiter by the Sun, must be similar to the perturbation of the Moon (the Earth's satellite) by the Sun. But in the latter case, we had (see pp. 239, 240.) about thirty terms to represent the effect of the Sun's disturbing force; in the present case we have only one: the terms

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$$* \; \frac{M}{n} = \frac{1^{\mathrm{d}}.7691378}{4332^{\mathrm{d}}.596308}.$$

depending on the eccentricities and inclination not appearing, because (see p. 360.) no account is made of those quantities.

The term, however,

$$\frac{11}{8}\,\frac{M^2}{n^2}\,\text{sin.}\,[2\,n\,t + 2\,\epsilon - (2\,M\,t + 2\,E)],$$

which does enter into the value of $\delta\,v$, and which is independent of the eccentricity, must be analogous to that principal term of the expression for the Moon's perturbation which is also independent of the eccentricity; and, in fact, it is analogous to the term expounding the *Lunar Variation*, which (see p. 239.) is

$$35'\ 46''\,.\,\text{sin.}\ 2\,(☽ - ⊙),$$

the argument of which, $2(☽ - ⊙)$, corresponds to $2\,n\,t + 2\,\epsilon - (2\,M\,t + 2\,E)$, or, twice the mean angular distance of Jupiter and his first satellite. The coefficient* of the *Satellite's Variation* (or rather of its principal term) is only $0''.047$. The other terms, therefore, are altogether insignificant.

The terms also which would represent, if the eccentricities of the orbits of the satellite and Jupiter were introduced, equations like the *Evection* and *Annual Equation*, are, when numerically expounded, too minute to be worth considering. The perturbations therefore of a satellite of Jupiter by the Sun, which, if we regard merely their mathematical and symbolical exposition, are the same as, and therefore equally difficult with, the Lunar, are in point of practice, exceedingly simple: and, they become simple, because the minute quantities of the eccentricities combined with that of the Sun's disturbing force annul, or render insignificant, all terms but one.

The Sun's disturbing force, then, has little to do with the inequalities of the first satellite. They are derived from the other satellites; but principally from the second; and of the terms expounding its perturbation (see p. 361.) it is only the fourth, or, rather, part of the fourth that claims much attention: for since, as it is found by observation, $n = 2\,n'$ nearly, the divisor $4\,(n' - n)^2 - n^2 = (2\,n' - n)\,(2\,n' - 3\,n)$ becomes very small, so that

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* The coefficient is $\frac{11}{8}\,.\,\frac{M^2}{n^2}$: for the second and third satellites, the coefficients are respectively $\frac{11}{8}\,.\,\frac{M^2}{n'}$, $\frac{11}{8}\,.\,\frac{M^2}{n''^2}$, and since $n = 2\,n'$, $n' = 2\,n''$, nearly, their numerical values will be $.047 \times 4$, $.047 \times 16$, respectively.

$$\frac{1}{2} \cdot \frac{m' n}{n - n'} \cdot \frac{2\,n^2}{4 \cdot (n - n')^2 - n^2} \left(a^2 . \frac{d\,\Gamma}{d\,a} + \frac{2\,n}{n - n'} . a\Gamma \right),$$

becomes a coefficient exceedingly greater, not only than the coefficients of the terms expounding the perturbations of the third and fourth satellite, but than the coefficients of the other terms due to the perturbation of the second.

Since $n = 2\,n'$, nearly, $4 \cdot (n - n')^2 - n^2 = (n - 2\,n')\,(3n - 2\,n') = (n - 2n').2n$, nearly, and, accordingly, $\frac{m'n}{2.(n - n')} \cdot \frac{2\,n^2}{4.(n - n')^2 - n^2} = \frac{m'n}{n - 2\,n'}$, nearly: let $F = a^2 . \frac{d\,\Gamma}{d\,a} + \frac{2\,n}{n - n'} . a\,\Gamma$, then,

$$\delta\,v = \frac{m'\,n\,F}{n - 2\,n'} \text{ sin. } 2\,(n'\,t - n\,t + \epsilon' - \epsilon),$$

is an expression nearly representing the inequality in longitude of the first satellite from the action of the second. And, indeed, this is the only inequality of the first satellite which has hitherto been discerned by observation.

As in the expression for δv, so in that for $\frac{r\,\delta\,r}{a^2}$, and for the same cause, there will be one term much larger than all the others; '*une terme dominante*,' as it is called by Bailly, who first, on Newton's principles and by Clairaut's methods, investigated the Theory of Jupiter's Satellites: this term will be (see p. 360.)

$$\frac{m'\,n^2}{4.(n - n')^2 - n^2} \cdot F \text{ cos. } (2\,n'\,t - 2\,n\,t + 2\,\epsilon' - 2\,\epsilon)$$

$$= \frac{m'\,n}{2(n - 2\,n')} \cdot F \text{ cos. } (2\,n'\,t - 2\,n\,t + 2\,\epsilon' - 2\,\epsilon).$$

These terms then, like the terms expounding the *great Inequality* of Jupiter and Saturn, derive their peculiarity from the near commensurability of the mean motions of the disturbed and disturbing satellites: but they have, besides, this distinction: they are, of all the terms representing the satellite's inequality, by far the largest; so that, if alone retained, they would adequately represent it. The terms, on the contrary, in the case of Jupiter and Saturn, are very small; much smaller than many other of the terms. They in fact, represent inequalities originating from a very small modification of the disturbing force, acting, however, for a long period. The inequalities of the first satellite, on the contrary, are the results of large modifications of the disturbing forces acting, with the same intent, during a considerable period.

The second satellite is subject, from the action of the first, to an inequality of the same kind as that which it causes to the first. The inequality is so eminent above the rest that one term suffices to represent it. Its period, however, is different from that of the former inequality, and it does not depend on the same argument. This will immediately appear from an examination of the analytical expression for $\delta v'$.

We shall obtain the expression for $\delta v'$, by writing in that for δv, m instead of m', n instead of n', n' instead of n, &c. in which case (see p. 308.)

$$\delta v' =$$

$$\left(5k' - \frac{7}{4}\cdot\frac{M^2}{n'^2} + \text{\&c.}\right) n't$$

$$+ \frac{11}{8}\cdot\frac{M^2}{n'^2}\sin.(2n't - 2Mt + 2\epsilon' - 2E)$$

$$+ \frac{mn'}{n'-n} \times$$

$$\left\{\begin{array}{l} \dfrac{n'}{n'-n}\cdot a'B' + \dfrac{2n'^2}{(n'-n)^2 - n'^2}\left(a'^2\cdot\dfrac{dB'}{da'} + \dfrac{2n'}{n'-n}a'B'\right)\sin.(nt - n't + \epsilon - \epsilon') \\ + \dfrac{1}{2}\left[\dfrac{n'}{n'-n}a'\Gamma' + \dfrac{2n'^2}{4.(n'-n)^2 - n'^2}\left(a'^2\cdot\dfrac{d\Gamma'}{da'} + \dfrac{2n'}{n'-n}a'\Gamma'\right)\right]\sin.2(nt - n't + \epsilon - \epsilon') \\ + \text{\&c.} \end{array}\right\}$$

Now in the preceding case it was a term in the fourth line of the value of δv that was rendered large by the divisor $n - 2n'$; but, in the present case, the divisor $4(n-n')^2 - n'^2 = (2n - 3n')(2n - n')$, neither of which factors is small: a term, however, in the third line receives the divisor $(n - n')^2 - n'^2 = (n - 2n')n$ the first of which factors is very small. The *predominant* term, therefore, in the expression for $\delta v'$ is

$$\frac{mn'}{n' - n} \cdot \frac{2n'^2}{(n - 2n')n} G . \sin. (nt - n't + \epsilon - \epsilon'),$$

(G being a quantity similar to F: see p. 363.), or,

$$- \frac{m'n'}{n - 2n'} G . \sin. (nt - n't + \epsilon - \epsilon').$$

The argument, therefore, of the large inequality of the first satellite, is

$$2(n't - nt + \epsilon' - \epsilon),$$

and, of the large inequality of the second caused by the perturbing force of the first, it is,

$$n't - nt + \epsilon' - \epsilon.$$

But the mean motion of the second satellite is nearly double that of the third. There must arise, therefore, from the action of the third, an inequality in the motion of the second precisely similar to that which the first receives from the second. It must depend on a similar argument and be expounded by a similar term: by a term, in fact, similar to $\frac{m'n}{n - 2n'} F . \sin. 2(n't - nt + \epsilon' - \epsilon)$, and which may be represented by

$$\frac{m''n'}{n' - 2n''} . F' . \sin. 2(n''t - n't + \epsilon'' - \epsilon'),$$

and this, as in the two former cases, is the paramount or predominant term, which, by itself, is adequate to represent the inequality in the longitude of the second satellite proceeding from the action of the third.

The inequality in the longitude of the third satellite from the action of the second must be similar to the inequality in the longitude of the second from the action of the first, and, accordingly, may be represented (see p. 365.) by

$$- \frac{m' n''}{n' - 2n''} G'.\sin.(n't - n''t + \epsilon' - \epsilon'').$$

The sum of the inequalities of the second satellite, produced by the disturbing forces of the first and third, is

$$\frac{m'' n'}{n' - 2n''} F'.\sin.2(n''t - n't + \epsilon'' - \epsilon') - \frac{m n' G}{n - 2n'} \sin.(nt - n't + \epsilon - \epsilon').$$

Now, (and this is the curious circumstance attending the inequality of the second satellite) the two arguments may be reduced to one: and this happens from a remarkable relation found by observation to exist between the mean longitudes of the three first satellites. It is this; *the mean longitude of the first Satellite minus three times that of the second plus twice that of the third is equal to* 180°; in symbols, then,

$$(nt + \epsilon) - 3(n't + \epsilon') + 2(n''t + \epsilon'') = 180^\circ,$$

consequently,

$$2n''t - 2n't + 2\epsilon'' - 2\epsilon' = n't + \epsilon' - nt - \epsilon + 180^\circ,$$

and (see *Trig.* p. 28.)

$$\sin.(2n''t - 2n't + 2\epsilon'' - 2\epsilon') = \sin.(nt - n't + \epsilon - \epsilon'),$$

the preceding variation ($\delta v'$), therefore is equal

$$\left(\frac{m''n'}{n' - 2n''} F' - \frac{mn'}{n - 2n'} G\right) \sin.(nt - n't + \epsilon - \epsilon'),$$

or, since $n' - 2n''$ nearly equals $n - 2n'$, is equal to

$$\frac{n'}{n - 2n'} (m''F' - mG) \sin.(nt - n't + \epsilon').$$

This, then, is the peculiar circumstance in the theory of the perturbations of the satellites to which we alluded in p. 356. The two inequalities which the second satellite receives from the disturbing forces of the first and third are blended together and appear as one inequality, having for its argument the mean angular distance of the first and second satellite. Something resembling this took place when the Moon was observed solely in eclipses (see *Astron.* p. 325.) for then the *Evection* and *Equation of the Centre* were confounded together: their arguments, in such a position, appearing to be the same. But the coefficient determined from such observations, was afterwards separated into its two parts, when the Moon was observed in other positions than those of opposition and conjunction. But this cannot be done in the present case. Whatever be the position of the third satellite, the argument of the inequality of the second is always the same, namely, the mean angular distance of the first and second. We cannot, therefore, by mere observation alone, separate into its two parts, one due to the action of the first satellite, the other to the action of the third, the coefficient, or the greatest value, of the inequality of the second satellite.

The inequalities which the fourth satellite causes and experiences are very minute. Its mean motion is not commensurable, nor nearly so, with the mean motion of the third satellite; and, consequently, not commensurable with the mean motion either of the second or third. There is, therefore, no predominant term to expound its inequality.

It is, however, most affected by the Sun's action. This is proved (see pp. 361, 362.) by the inspection of the term that expounds that equation which we have considered as analogous to the Lunar Variation: but, on the plainest principles, it is clear that the farther the satellite is removed from its primary, the less must be the attractive force of the latter compared with the Sun's; and the greater must be the Sun's disturbing force, the more inclined the direction of his action becomes.

The greatest value of the variation of the fourth satellite of Jupiter is nearly 4″.2. Newton, in the twenty-third Proposition of the third Book of his *Principia*, makes it 5″ 12‴.

The argument of a satellite's variation produced by the Sun's disturbing force is twice his mean angular distance from the Sun. If, analogously to the use of ☽, ☉ ♃, &c. in the Lunar and Planetary Theories (see pp. 217, 310.) we employ ☾′, ☾″, &c. to represent the mean motions of the first, second, &c. satellites of Jupiter, the arguments of their respective variations will be

$$2.(\odot - ☾'),\ 2.(\odot - ☾''),\ 2.(\odot - ☾'''),\ 2.(\odot - ☾^{\text{iv}}),$$

and the coefficients

$$\frac{11\,M^2}{8\,n^2} = .04729,\quad \frac{11\,M^2}{8\,n'^2} = .19054,$$

$$\frac{11\,M^2}{8\,n''^2} = .77338,\quad \frac{11\,M^2}{8\,n'''^2} = 4''.20814.$$

The Lunar *Variation* (see p. 219.) is represented by several terms. The *Variations* of the satellites, if we look merely to theory, ought to be represented by as many; but, considering the smallness of the largest of their coefficients, they are each adequately represented by a single term of which the argument is twice the mean angular distance of the Sun and satellite. The largest term in the Lunar Variation has a similar argument: a correspondence which must needs happen, seeing the very strict analogy between the Moon's perturbations by the Sun, and a satellite's perturbation by the Sun.

But the analogy between the Lunar perturbations and the mutual perturbations of the satellites is so vague that if we took it for a guide it would lead us wrong. The variation indeed of the first satellite caused by the disturbing force of the second has for its argument $2(☽' - ☽'')$: but the argument of the variation of the second satellite caused by the first is $☽' - ☽''$, or, rather, the above angular distances are respectively the arguments of the principal terms. These remarks are of the same tenor with those that are stated at p. 311.

The mutual perturbations of the first and third satellite are very inconsiderable: but both these disturb, and sensibly, the motion of the second. In order to determine the period of the inequalities of the three first satellites, we have to observe, that they depend on their relative situation. When, at the end of any interval, the three satellites return to the same relative situation, or technically, have the same *configuration* which they had at the beginning, then such interval must be the *period* of the inequalities. Its value admits a theoretical investigation: but, without the aid of any such investigation, Bradley ascertained it. Indeed it could not easily escape detection when the synodic revolutions (and these are the revolutions that observation determines) were scrutinised: thus, the synodic periods of the three first satellites are

$1^d\ 18^h\ 28^m\ 36^s$, or in decimals, $1^d.769860$
3 13 17 54 .... 3.554090
7 3 59 36 .... 7.154579:

consequently, as we find by the use of *continued fractions* *, 247 revolutions of the first satellite are absolved in nearly the same time as 123 of the second: for the numbers 123, 247, are to each other very nearly in the proportion of 1.76986 to 3.55409. The proportion is not exact; 247 revolutions of the first satellite, however, being completed in $437^d\ 3^h\ 44^m$, and 123 of the second in $437^d\ 3^h\ 41^m$, we may assume, with very little inaccuracy, 437 days as the period for the return of the two first satellites to their original *configuration*. But, besides, 61 revolutions of the third satellite being absolved in $437^d\ 3^h\ 35^m$, the same period of 437 days †, is that, at the end of which the three satellites will have

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* If we divide 3.55409 by 1.76986, the latter by the remainder, and so on (see Wood's *Algebra*, on continued Fractions) the series of quotients will be

2, 123, 6, 8, &c.

consequently,

$$\frac{1}{2},\ \frac{123}{247},\ \frac{739}{1482},\ \&c.$$

are the fractions, which according their order, are alternately greater and less than, but successively nearer to, the true value of $\frac{1.76986}{3.55409}$.

† $437^d.659$ is the exact period.

the same configuration as they had at the beginning; and, consequently, it is the period during which the three satellites must have passed through every inequality, as well in kind as degree, that arises from their mutual perturbation.

From the rapidity then of the revolutions of the satellites, their inequalities, even those which have the longest period, quickly recur. In one respect, then, their theory admits a more complete verification than the theory of the planets: for, it must be left to times that are to come, the establishing by observation of that inequality of Jupiter and Saturn which has for its period more than 918 years *.

The inequalities, which we have considered, are independent of the eccentricities and inclinations of the orbits: they depend solely on the mean elongation of the disturbed and disturbing satellites, or on multiples of that elongation, and are, as it has been remarked, a species of *variation.* They serve to explain why an eclipse of a satellite happens sooner or later than it ought to do either according to the circular, or Kepler's Elliptical Theory: they will serve, therefore, to perfect the Tables of the Satellites' motions: and, under another point of view, they serve, by explaining the retardation of an eclipse on Newton's principles, to illustrate and confirm them.

Almost all the materials for forming Tables of the motions of the Satellites are drawn from the epochs and durations of their eclipses. The elongations of the satellites serve to determine the masses of their primaries and the mean distances from their centres. But the mean distance of one satellite being once accurately determined the mean distances of the others are best determined, not from their elongations, but by means of Kepler's Law. In this case computation is far superior to direct observation: and mainly for this reason; the period of the satellite from which its mean distance is computed, is itself not determined wholly and immediately by observation, but through the intervention of computation: for instance, the sidereal period is deduced from the synodic: the synodic from observations of the satellite in conjunction with Jupiter: either on the disk of the primary or in his shadow. Now, whatever be the error in noting one synodic period, or in noting the interval between an eclipse and the next succeeding one, there need not be a greater error in noting the interval between the first eclipse and an eclipse the hundredth from it. The synodic period, therefore, determined by dividing the interval between two remote eclipses by the number (plus one) of intervening eclipses must be very exact, from the diminished effect of the incidental or probable error of observation *. The like to this cannot take place with observations of elongations: if we would know what an hundred would amount to, we must needs measure every one.

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* See on this subject, *Mem. Acad.* 1788, p. 271. Also *Mec. Cel.* 2d Part. Liv. VIII. p. 18: where M. Laplace determines the period on different principles.

Those elements, therefore, the mean distances and mean motions, are easily determined. But in the preceding formulæ there are arbitrary symbols involved, which, except they can be numerically exhibited, would render nugatory the whole theory of the perturbations of the Satellites. We mean the symbols m, m', m'', &c. which denote the masses. There are certainly no obvious means of determining them. A satellite of Jupiter is indeed his Moon, but with this peculiarity of additional difficulty, when the question is concerning its quantity of attraction (which is to expound its mass) that there are, for that purpose, no phenomena of the tides of the primary planet. Other phenomena of attraction must, therefore, be looked to. But the difficulty we have spoken of is not peculiar to the theory of the perturbations of Jupiter's satellites: it occurs whenever the disturbing planet is one that has no satellite; consequently, when either Mercury, or Venus, or Mars, is the disturbing body. If, therefore, we would perfect the Solar and Planetary Tables, or, on another ground, if we would put Newton's Theory to the severest test, we must determine the masses of those planets. The necessity or importance

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* There is in this method a principle somewhat akin to that which prevails in Borda's Repeating Circle.

of such an enquiry is now only adverted to. The enquiry itself will be entered on in a subsequent Chapter. But, at present, the natural course of investigation is towards other points.

It is now to be directed towards those points which before (see pp. 252, &c.) were approached and noticed: namely, the elements of a planet's orbit, or those quantities on which its dimensions and position in space depend. The periodical inequalities of the Moon, the Planets, and Jupiter's satellites have been computed from the disturbing forces: and we have now to examine what effect those same disturbing forces produce on the mean distances and eccentricities which determine the dimensions of orbits; and on the nodes, apsides and inclinations, quantities on which the positions of orbits in space depend. It is not likely, viewing the matter theoretically, that these elements should remain constant: for they are so in the elliptical or undisturbed system: and from phenomena we certainly know they are variable: at least that four of them are; the nodes, the apsides, the inclination by direct observation, the eccentricities by inference from observation. With regard to the mean distance; by inference from observation, that element remains invariable; for, the mean motion, whether of the Moon, or of a planet, is not found to change. The strict inference, indeed, is that it is not subject to a *secular* change. But even this kind of invariability is a remarkable circumstance. It is, if we may so call it, a *phenomenon* well deserving explanation: and it will be, if explained by Newton's Theory, not one of the least curious of its results. It is, indeed, a theoretical result not to be anticipated, that the disturbing forces, which impair the equable description of areas, the regularity of a planet's orbit, and alter its position in space, should still leave unchanged the dimensions of the orbit.

The variation of the inclination of a planet's orbit, the variations of the longitudes of its node, and aphelion, or technically, *Regression* of the Node, and the *Progression* of the Aphelion * were known phenomena at the time of the invention of Physical Astronomy. It was important, therefore, for Newton to shew that these phenomena, as results from theory, were parts of his system. This he has done with regard to the two first. In his eleventh Section and in the thirtieth, thirty-first, thirty-second, thirty-third, thirty-fourth Propositions of the third Book, Newton has shewn that the changes of the plane's inclination, and of the place of the node, are necessary effects of the disturbing force: and, besides, he has computed those effects. Nothing can be more simple, direct, and perspicuous, than the method used in these Propositions. But, unfortunately, the method with its excellences here terminates. There is no similar method in the *Principia*, nor in any other Work, for determining the progression of the apsides. That variation, as an object to be sought after, is more subtle in its nature, than the preceding: it eluded, as it is well known, Newton's research; and remained a *desideratum* in Physical Astronomy, till, about the year 1748, Clairaut, Dalembert, and Thomas Simpson, began to cultivate that science.

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* Not always a *progression*.

These mathematicians (see Chapters IX. X. XI. XIII.) computed the progression of the Lunar Apogee on Newton's Principle and Law of Gravity, and shewed that its computed agreed with its observed quantity. The method, however, which they employed, is not a direct and obvious one, and assigns only the mean quantity of the progression. If we were to join this latter method to Newton's, we should have, for determining the variations of three of the elements, two methods totally dissimilar. Their mere dissimilarity, however, in itself is not a sufficient reason for rejecting them. We may find out and adopt others related, indeed, but without any other claim to preference. If, however, we adopted Newton's method for the nodes and Clairaut's for the apsides, we should still have to seek methods for the variations of the eccentricity and axis major: and none would be found to present themselves resembling either the one or the other of the preceding methods.

So it happens, however, and the circumstance is to be accounted for. The things to be investigated, are, it may be said, in their nature, dissimilar. As each element is separately considered, a peculiar method of estimating the effect of the disturbing force on it is suggested. It is not, therefore, by so considering the subject that a general method can be obtained. We must view the subject differently.

We must, for a time, disregard the peculiar nature of each element and find out or feign conditions of general resemblance. Now, this may be effected on the principles laid down in pp. 39, &c. The arbitrary quantities introduced by integration are to be considered as the elements of the orbit described, and, consequently, the differentials of such quantities will represent the variations of the elements.

The differentials may, as in other cases, be found by the ordinary rules; but there is, in the subject we are now speaking of, some difficulty in reducing them to similar formulæ. Mere similarity, indeed, is in itself not necessarily an advantage: but the similar formulæ just alluded to will be found most convenient; both for the making of general inferences and for illustration by examples.

This is the mere suggestion of the plan which will be more fully developed in the ensuing Chapter.

CHAP. XXI.

ON THE VARIATIONS, PERIODICAL AND SECULAR, OF THE ELEMENTS OF THE ORBITS OF PLANETS.

Principle of the Method for determining the Variations of the Element's of a Planet's Orbit. The Elements viewed as the Arbitrary Quantities introduced by the Integration of the Differential Equations of Motion, or as their Functions. Expressions for the Variations of the Mean Distance, the Eccentricity and the Longitude of the Perihelion: the Variation of the Eccentricity expressed by means of partial Differential Coefficients of the Quantity (R) *dependent on the Disturbing Force: the same Form of Expression extended to the Variations of the other Elements. The Origin and the Authors of these Expressions.*

If we revert to pp. 39, &c. we shall perceive that, according to the *analytical view* there taken of the subject, the arbitrary quantities, introduced by the integration of the differential equations, are either the elements of a planet's orbit, or functions of those elements.

They are the elements themselves, when the equations are expressed in terms of the projected radius vector, the longitude and the tangent of latitude: that is, when the three equations (abstracting the disturbing forces), are

$$d^2\rho - \rho\, dv^2 + \frac{dt^2}{\rho^2.(1+s^2)^{\frac{3}{2}}} = 0,$$

$$\rho\, d^2 v + 2\, d\rho\, dv = 0,$$

$$d^2(\rho s) + \frac{s}{\rho^2(1+s^2)^{\frac{3}{2}}} = 0,$$

and functions of the elements, when, by means of the rectangular co-ordinates x, y, and z, the equations are thus symmetrically expressed,

$$\frac{d^2 x}{d t^2} + \frac{\mu x}{r^3} = 0,$$

$$\frac{d^2 y}{d t^2} + \frac{\mu y}{r^3} = 0,$$

$$\frac{d^2 z}{d t^2} + \frac{\mu z}{r^3} = 0.$$

Both these sets of equations (for they may be mutually derived, the one from the other) belong to the elliptical system, consisting only of two bodies, and in which there is no disturbing force. As we have already seen (pp. 33, &c.) the elements of the elliptical system, the axis major, eccentricity, &c. are invariable. The arbitrary quantities, introduced by the integration of the differential equations, are constant arbitrary quantities; and, whilst the equations retain the preceding form, they are capable of an exact integration. But there is no case in the planetary theory to which the preceding equations exactly apply. The elliptical laws of form and revolution are never strictly observed: there is always, more or less, some disturbance: and the preceding equations, before they are applicable to the planetary theory, require the addition of certain small terms dependent on the disturbing force. For instance, the equation of l. 1. requires the addition of $\frac{dR}{dx}$.

$$\left(R = m' \frac{x x' + y y' + z z'}{r'^3} - \frac{1}{\lambda} \quad \text{see p. 68.}\right)$$

and then it becomes

$$\frac{d^2 x}{d t^2} + \frac{\mu x}{r^3} + \frac{dR}{dx} = 0,$$

and the two other equations require, respectively, the addition of $\frac{dR}{dy}$, and $\frac{dR}{dz}$.

$\frac{dR}{dx}$, $\frac{dR}{dy}$, $\frac{dR}{dz}$, which expound the disturbing force, are very small, and therefore the integration of the differential equations, from which these terms are rescinded, must be *nearly* the integration of the equations when they are complete. The constant arbitrary quantities also, determined for the former equations, although they cannot remain the same, or retain their property of invariability, yet they cannot be subject to any except small variations, since the two sets of equations differ only by very small terms such as $\frac{dR}{dx}$, &c.

The mean distance, for instance, if it were possible to determine it by the integration of the *complete* equations, could only differ, by a very small quantity, from that value of the mean distance which results from the actual integration of the imperfect equations of p. 376. This small difference, whatever be its expression, between the two mean distances, must be dependent on the disturbing force; for, the two sets of equations differ only by those small terms which would be nothing were there no disturbing force. Hence, (and it was by reasoning nearly in this way that the method was arrived at) the expressions for the constant arbitrary quantities resulting from the integration of the elliptical equations may be assumed as the expressions for the variable arbitrary quantities, on the condition of determining the variations of the latter from the differences between the two sets of equations. For, neither do the arbitrary quantities vary, nor do the equations differ, except by reason of the disturbing force. Suppose, then, *a* to be an arbitrary quantity, and that, by the integration of the *elliptical* equations, or of an equation resulting from their combination, we obtain an equation of the first order, such as

$$V = a.$$

V involving, or being a function of, x, y, z, $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$. If in dV we substitute for $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2z}{dt^2}$, the quantities $\frac{-x}{r^3}$, $\frac{-y}{r^3}$, $\frac{-z}{r^3}$, or those values which result from

$$\frac{d^2 x}{d t^2} + \frac{x}{r^3} = 0 \ldots\ldots\ldots\ldots\ldots (\mu = 1),$$

$$\frac{d^2 y}{d t^2} + \frac{y}{r^3} = 0,$$

$$\frac{d^2 z}{d t^2} + \frac{z}{r^3} = 0,$$

then there must result the identical equation,

$$d V = 0.$$

But if $V = a$ be assumed an integral equation of an equation formed by combining the equations,

$$\frac{d^2 x}{d t^2} + \frac{x}{r^3} + \frac{d R}{d x} = 0,$$

$$\frac{d^2 y}{d t^2} + \frac{y}{r^3} + \frac{d R}{d y} = 0,$$

$$\frac{d^2 z}{d t^2} + \frac{z}{r^3} + \frac{d R}{d z} = 0,$$

then in $d V = d a$, we cannot, as before, substitute $\frac{-x}{r^3}$ instead of $\frac{d^2 x}{d t^2}$, $\frac{-y}{r^3}$ instead of $\frac{d^2 y}{d t^2}$, &c. since the values of $\frac{d^2 x}{d t^2}$, $\frac{d^2 y}{d t^2}$, &c. are different, as it is plain from the equations, or, as we may at once infer from this consideration; namely, that, the forces in the two cases are different, and forces are expounded by the second differentials or fluxions of quantities (see Preface to *Principles of Anal. Calc.* pp. 5, 6.). $\frac{d^2 x}{d t^2}$, $\frac{d^2 y}{d t^2}$, $\frac{d^2 z}{d t^2}$ are, then, symbols of different values in the two cases, and, if we suppose the symbol δ to denote the effect of the disturbing force, the first equation may be thus written,

$$\frac{1}{d t} \cdot d\left(\frac{d x + \delta x}{d t}\right) + \frac{x}{r^3} + \frac{d R}{d x} = 0,$$

$$\text{or, } \frac{d^2 x}{d t^2} + \frac{d \delta x}{d t^2} + \frac{x}{r^3} + \frac{d R}{d x} = 0.$$

But by the equations of p. 376.

$$\frac{d^2 x}{dt^2} + \frac{x}{r^3} = 0;$$

$$\therefore \frac{d\,\delta x}{d\,t^2} + \frac{d R}{d x} = 0.$$

Hence, in dV, if dx, dy, dz, and a alone are made to vary, it will be sufficient to write $-\frac{dR}{dx}$, $-\frac{dR}{dy}$, $-\frac{dR}{dz}$, instead of $\frac{d^2 x}{dt^2}$, $\frac{d^2 y}{dt^2}$, $\frac{d^2 z}{dt^2}$, and the resulting equation,

$$dV = da,$$

will give the value of da; or, the same result will be obtained if we use the symbol δ, and in

$$\delta V = \delta a$$

substitute $-\frac{dR}{dx}$, $-\frac{dR}{dy}$, $-\frac{dR}{dz}$, instead of $\frac{d\,\delta x}{dt^2}$, $\frac{d\,\delta y}{dt^2}$, $\frac{d\,\delta z}{dt^2}$.

An instance will illustrate the principle of the method; if we multiply the equations of p. 376, by dx, dy, dz, respectively, add them and integrate the equation so formed, there will result this integral equation,

$$\frac{dx^2 + dy^2 + dz^2}{2\,dt^2} - \frac{\mu}{r} + \frac{\mu}{2a} = 0,$$

in which, a is an arbitrary quantity introduced by integration. If we compare this with $V = a$, (see p. 378.)

$$\frac{dx^2 + dy^2 + dz^2}{2\,dt^2} - \frac{\mu}{r} \text{ corresponds to } V,$$

$$\text{and, } -\frac{\mu}{2a} \text{ to } a;$$

$$\therefore \frac{dx.d\delta x + dy.d\delta y + dz{:}d\delta z}{dt^2} = \delta V,$$

$$\text{and } \frac{\mu}{2} \cdot \frac{\delta a}{a^2} \text{ corresponds to } \delta a,$$

consequently, by the Rule

$$-dx.\frac{dR}{dx} - dy.\frac{dR}{dy} - dz.\frac{dR}{dz} = \frac{\mu}{2}.\frac{\delta a}{a^2}.$$

But if R be a function of x, y, z, the left-hand side of the equation is (see *Prin. Anal. Calc.* pp. 78, &c.) the complete differential of $-R$, which is usually thus expressed $-dR$; consequently,

$$-dR = \frac{\mu}{2}.\frac{\delta a}{a^2}.$$

This result has been obtained by means of the symbol δ, and of the process indicated by it; that is, (see p. 378.) by an abridgment of the direct and plainer method. This latter, however, in the present case, is easily instituted; thus, since

$$r = \surd(x^2 + y^2 + z^2),$$

$$dV = \frac{dx.d^2x + dy.d^2y + dz.d^2z}{dt^2} + \mu.\frac{x\,dx + y\,dy + z\,dz}{r^3};$$

but

$$\frac{d^2x}{dt^2} = -\mu\frac{x}{r^3} - \frac{dR}{dx},$$

$$\frac{d^2y}{dt^2} = -\mu\frac{y}{r^3} - \frac{dR}{dy},$$

$$\frac{d^2z}{dt^2} = -\mu\frac{z}{r^3} - \frac{dR}{dz},$$

substitute these values in dV, and there will result,

$$dV = -\left(dx.\frac{dR}{dx} + dy.\frac{dR}{dy} + dz.\frac{dR}{dz}\right),$$

whence the same result as before *.

In the other process indicated by

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* If dx, dy, dz, only are made to vary and not r, the last term in dV (l. 12.) may be suppressed as well as the second terms in ll. 14, 15, 16. and the same result will still subsist; which is the first part of the rule of p. 379.

$$\delta V = \delta a,$$

the expression $\frac{x\,\delta x + y\,\delta y + z\,\delta z}{r^3}$, which would result from $-\delta\left(\frac{1}{r}\right)$, was suppressed, since δx, δy, δz, are equal nothing; for, the differential equations of the first order are the same both in the undisturbed and disturbed system: and since the differentials of x, y, z, when the disturbing force acts, have been expressed by

$$dx + \delta x, \quad dy + \delta y, \quad dz + \delta z,$$

we must necessarily have

$$\delta x = 0, \quad \delta y = 0, \quad \delta z = 0,$$

which, in fact, are three equations of condition between the variations of the arbitrary quantities; for instance, if x contain two arbitrary quantities a and b, then

$$\delta x = \frac{dx}{da}\,\delta a + \frac{dx}{db}\,\delta b = 0,$$

which equation determines the relative values of δa and δb.

Hence, the abridged method, under its most simple form, is made to depend on equations such as these

$$\frac{d\,\delta x}{dt^2} = -\frac{dR}{dx},$$

$$\delta x = 0.$$

We have used the symmetrical equations of p. 376. and the equations derived from them, chiefly for illustration: but the method applies equally to any equations derived from the preceding. In deducing the variations of the other elements, it is not intended to use the symmetrical equations; because, as it has been observed, their integration affords not simply the elements of a planet's orbit, but functions of those elements.

In the instance used for exemplifying the equation $V = a$ (see p. 379.) *it so happens* that the quantity a therein introduced, is the mean distance, one of the elements. But this circumstance is peculiar to that differential equation. The five other equations may be constructed so as to involve each only one arbitrary quantity; but, then, such arbitrary quantity will not be an element.

The three equations of p. 376. being of the second order admit, each, of two integrations: each integration introduces an arbitrary quantity, therefore, there will be six differential equations of the first order (like the one of p. 378.) and six * arbitrary quantities.

Any equation such as $V = a$, of the first order belongs equally (see p. 381.) to the disturbed and undisturbed system. Hence, to the same systems the integral of that equation must also belong. The finite expressions, therefore, of x, y, z are the same: but there is this distinction to be noted in their values; in the latter, which is the elliptical system, the arbitrary quantities are constant, whilst in the former they are variable, their variations being determined by formulæ similar to that already obtained. Now, since the expressions for x, y, and z are the same, the curves of which these are the co-ordinates must be similar, or of the same kind; but, in the undisturbed system, the curve is an ellipse: the curve therefore to which x, y, z are the co-ordinates, when the arbitrary quantities a, b, c, &c. are variable, must be also an ellipse: this, however, is not the curve described by the body: it is merely the ellipse that would be described were the disturbing forces to cease at that point of time for which the arbitrary quantities a, b, c, &c. were determined. At the next instant a, b, c, &c. have different values, and the ellipse is of different dimensions: so that, as it is easy to see, the successive ellipses form a series of *ellipses of curvature* to the real curve.

It is easy to see that the method which has been described rests on the same principle as that which is technically denominated the *Variation of the Parameters*, and which was employed in pp. 96, &c.

We will now proceed to deduce the variations of the elements on the principles already laid down, but by the aid of those equations which involve the projection of the radius vector, the longitude and the tangent of the body's latitude: these equations are (see pp. 92, 66.)

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* These are not independent, the one of the other; the equation which connects them reduces their number to five.

$$\rho . d^2 v + 2 d\rho . dv + \frac{1}{\rho} . \frac{dR}{dv} dt^2 = 0 \ldots\ldots\ldots\ldots (1),$$

$$d^2\rho - \rho . dv^2 + \left(\frac{\mu}{\rho^2 . (1 + s^2)^{\frac{3}{2}}} - \frac{s}{\rho} . \frac{dR}{ds} + \frac{dR}{d\rho}\right) dt^2 = 0 \ldots . (2),$$

$$d^2(\rho s) + \left(\frac{\mu s}{\rho^2 . (1 + s^2)^{\frac{3}{2}}} + \frac{1}{\rho} . \frac{dR}{ds}\right) dt^2 = 0 \ldots\ldots\ldots (3),$$

and, if we rescind from these the terms $\frac{dR}{d\rho}$, $\frac{dR}{dv}$, $\frac{dR}{ds}$, which depend on the disturbing force, there will remain three equations for determining the elliptical laws of the body's motion.

If we appropriate, as in p. 378. the symbol δ to represent the effects of the disturbing force, then when that force acts

$$d^2 v = d(dv + \delta v) = d^2 v + d\delta v,$$
$$d^2\rho = d(d\rho + \delta\rho) = d^2\rho + d\delta\rho.$$

Substitute these values in the two first equations, and there will result, by virtue of the equations of condition,

$$\delta\rho = 0, \quad \delta v = 0,$$

and of the *elliptical* equations mentioned in l. 9.

$$\rho . d\delta v = - \frac{1}{\rho} . \frac{dR}{dv} dt^2,$$

$$d\delta\rho = \left(\frac{s}{\rho} . \frac{dR}{ds} - \frac{dR}{d\rho}\right) . dt^2,$$

or, if we suppose the latitude s to equal nothing, and r to be the radius vector,

$$d\delta v = - \frac{dR}{r^2 . dv} . dt^2,$$

$$d\delta r = - \frac{dR}{dr} . dt^2,$$

and, in this case, v is the body's longitude measured in the plane of the orbit.

The two equations (1), (2), when the disturbing force is rescinded, and ρ becomes r, are

$$r\,d^2 v + 2\,d r . d v = 0,$$

$$d^2 r - r\,d v^2 + \mu \frac{d t^2}{r^2} = 0,$$

multiply the first of these by $\frac{r\,d v}{d t^2}$, and add it to the second multiplied by $\frac{d r}{d t^2}$, then there results

$$\frac{r^2\,d v\,d^2 v + r\,d r\,d v^2 + d r . d^2 r}{d t^2} + \frac{\mu\,d r}{r^2} = 0,$$

and the integral of this is

$$\frac{r^2\,d v^2 + d r^2}{2\,d t^2} - \frac{\mu}{r} + \frac{\mu}{2 a} = 0,$$

a differential equation of the first order similar to the one of p. 379, and obtained by similar means, and which may be similarly used for determining the variation of the arbitrary quantity a which is introduced by integration: we have then by the rule (see p. 379, &c.)

$$\frac{2\,r^2\,d v\,d \delta v + 2\,d r\,d \delta r}{2\,d t^2} + \frac{\mu}{2}\,\delta\left(\frac{1}{a}\right) = 0,$$

but, see p. 383.

$$\frac{d \delta v}{d t^2} = -\frac{1}{r^2} . \frac{d R}{d v},$$

$$\frac{d \delta r}{d t^2} = -\frac{d R}{d r};$$

$$\therefore\ d v . \frac{d R}{d v} + d r . \frac{d R}{d r} = \frac{\mu}{2}\,\delta\left(\frac{1}{a}\right) = \frac{-\mu}{2} . \frac{\delta a}{a^2}.$$

Now, when s is 0, R is a function of r and v, and the left-hand side of the equation is the complete differential of R: therefore, as before,

$$\delta a = -\frac{2a^2}{\mu}.dR.$$

The quantity a is (see p. 379.) the mean distance, or the semiaxis major of the ellipse. The first result then, and indeed the easiest of the method that has been explained, is the expression of the variation of the axis major. We will soon attend to the remarkable consequence that may be deduced from that expression.

If we multiply the equation (1) (writing r instead of ρ) by r, we have

$$r^2 d^2 v + 2 r\, d r\, d v + \frac{d R}{d v} d t^2 = 0.$$

But, $r^2 d^2 v + 2 r\, dr\, dv = d(r^2 d v) =$ (see p. 14.) $d(h\, d t)$.

Hence, $\quad dh.dt + \dfrac{dR}{dv} dt^2 = 0,$

and $\quad dh$, or, $\delta h = -\dfrac{dR}{dv} dt.$

Now, see p. 25, h, the mean distance a, and the eccentricity e are connected together by this equation,

$$h^2 = \mu a.(1 - e^2) = a(1-e^2), \text{ if } \mu = 1.$$

Hence, $\quad 2h\delta h = \delta a.(1 - e^2) - 2ae\delta e,$

or $\quad -2h.\dfrac{dR}{dv} dt = -2a^2.(1 - e^2)dR - 2ae\delta e,$

whence,

$$\delta e = \frac{h}{ae}.\frac{dR}{dv} dt - \frac{a(1-e^2)}{e}.dR,$$

$$\text{or} = \frac{\sqrt{[a.(1-e^2)]}}{ae}.\frac{dR}{dv} dt - \frac{a.(1-e^2)}{e}.dR.$$

In order to deduce $\delta\pi$, we may use the equation of condition, namely,

$$\delta r = 0,$$

and this equation is like $\delta x = 0$ (see p. 381.) that is, is to be formed by taking the differential or fluxion of the value of r, supposing all those elements or arbitrary quantities, which in the elliptical or undisturbed system are constant, to vary. Now, (see p. 25.)

$$r = \frac{h^2}{1 + e \,.\, \cos.\,(v - \pi)};$$

$$\therefore \delta r = \frac{dr}{d.h^2}\,\delta h^2 + \frac{dr}{d\pi}\,\delta\pi + \frac{dr}{de}\,\delta e = 0.$$

But *,

$$\frac{dr}{dh^2} = \frac{1}{1 + e \,.\, \cos.\,(v - \pi)},$$

$$\frac{dr}{d\pi} = -\frac{e h^2 \sin.\,(v - \pi)}{[1 + e \,.\, \cos.\,(v - \pi)]^2},$$

$$\frac{dr}{de} = -\frac{h^2 .\, \cos.\,(v - \pi)}{[1 + e \,.\, \cos.\,(v - \pi)]^2}.$$

Substitute these values in the equation $\delta r = 0$, and instead of δh^2 $(= 2h \,.\, \delta h)$ and δe, the values (see p. 385. ll. 11, 18.) already obtained, and there will result

$$\delta\pi = -\frac{2e + (1 + e^2) .\, \cos.\,(v - \pi)}{h e^2 .\, \sin.\,(v - \pi)} \,.\, \frac{dR}{dv} \,.\, dt$$

$$+ \frac{h^2}{e^2} \cos.\,(v - \pi) . dR.$$

We have now three expressions, for the variations of the axis major, the eccentricity, and the longitude of the apogee; and these expressions are as convenient as any that can be exhibited, if the results are to be expressed in terms of v, &c. For (see p. 373.)

$$R = \frac{m' r}{r'^2} \cos.\,(v - v') - \frac{m'}{\sqrt{[r^2 - 2 r r' \cos.\,(v - v') + r'^2)}},$$

and thence,

$$\frac{dR}{dv}, \ \frac{dR}{dr}, \text{ and } dR \left(= \frac{dR}{dv} dv + \frac{dR}{dr} dr \right),$$

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* See *Principles of Analytical Calculation*, pp. 79, &c.

may easily be computed. In the Lunar theory, R is usually expressed by a series of cosines of multiples of v; but, in most of the cases that occur in the Planetary theory, R, by reason of the small eccentricities and inclinations, can be at once expanded into a series of cosines, involving, not v the true anomaly, but nt the mean (see Chapter XVII.). In such an expansion, then, the preceding expressions for the variations would not immediately be applicable. $\frac{dR}{dv}$, for instance, would be without significancy. It becomes necessary then, in order to adapt the preceding expressions to the usual mode of expressing R, to convert $\frac{dR}{dv}$ into some other partial differential coefficient of R; R being, in this latter case, a function of nt and of the elements a, e, π, &c.

Now, with regard to the first variation, that of a, no conversion is necessary: for,

$$\delta a = -2a^2 . dR,$$

in which dR is the complete differential of R. If R should be a function of r and v, then

$$dR = \frac{dR}{dr} dr + \frac{dR}{dv} dv,$$

if a function of ρ, v, and s, then

$$dR = \frac{dR}{d\rho} d\rho + \frac{dR}{dv} dv + \frac{dR}{ds} ds,$$

so that, if R should be transformed into a function of nt and of other but constant quantities, by converting (see p. 274.) r, v, or ρ, v, s into series of terms involving the sines and cosines of nt and other quantities, then, dR would be obtained by merely making nt to vary in the expression for R; for instance, if one of the terms representing R, should be

$$A . \cos. (int - i'n't + B),$$

the corresponding value of dR, would be

$$-inA . \sin. (int - i'n't + B).$$

With regard to the second variation, that of the eccentricity, some conversion is necessary: now,* if R be a function of v, and v of nt, then,

$$\frac{dR}{ndt} = \frac{dR}{dv} \cdot \frac{dv}{ndt}.$$

But, see p. 274. (using A, B, or the coefficients of the third and fourth terms),

$$v = nt + \epsilon + A . \sin. (nt + \epsilon - \pi)$$
$$+ B \sin. (2nt + 2\epsilon - 2\pi) + \&c.$$

$$\therefore \frac{dv}{ndt} = 1 + A . \cos. (nt + \epsilon - \pi) + \&c.$$

But

$$\frac{dv}{d\pi} = - A . \cos. (nt + \epsilon - \pi) - \&c.$$

$$\therefore \frac{dv}{ndt} = 1 - \frac{dv}{d\pi};$$

consequently,

$$\frac{dR}{ndt} = \frac{dR}{dv} - \frac{dR}{dv} \cdot \frac{dv}{d\pi}$$
$$= \frac{dR}{dv} - \frac{dR}{d\pi},$$

$$\text{or, } \frac{dR}{dv} = \frac{dR}{ndt} + \frac{dR}{d\pi}.$$

Hence, instead of the former expression for δe, we have this

$$\delta e = \frac{\surd[a(1-e^2)]}{ae} \left(\frac{dR}{ndt} dt + \frac{dR}{d\pi} dt\right) - \frac{a(1-e^2)}{e} . dR.$$

But dR (see p. 387. l. 16.) being the complete differential of R, and $= \frac{dR}{ndt} ndt$; and n being $= a^{-\frac{3}{2}}$;

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* See *Principles of Analytical Calculation*, pp. 89, 90.

$$\therefore \delta e = a \,.\, \frac{\sqrt{(1-e^2)}}{e} dR - \frac{a(1-e^2)}{e} d R + \frac{\sqrt{(1-e^2)}}{\sqrt{a \,.\, e}} \,.\, \frac{d R}{d \pi} d t,$$

$$\text{or} = .\, \frac{a\sqrt{(1-e^2)}}{e} .[1-\sqrt{(1-e^2)}]\, dR + \frac{\sqrt{(1-e^2)}}{\sqrt{a.e}} \,.\, \frac{d R}{d \pi} d t.$$

We may still vary the expression for δe, without obtaining, however, a more commodious one for computation. For, if we examine the expression for v (see p. 388. l. 6.), it will be seen that

$$* \ \frac{d R}{n\, d t} = \frac{d R}{d \epsilon},$$

since, in every part of that expression where $n t$ is, ϵ is also, hence †,

$$\delta e = \frac{\sqrt{(1-e^2)}}{\sqrt{a \,.\, e}} \left(\frac{d R}{d \epsilon} + \frac{d R}{d \pi}\right) d t - \frac{a(1-e^2)}{e} \,.\, \frac{d R}{d \epsilon} n\, d t$$

$$= \frac{a}{e} \sqrt{(1-e^2)} \left(\frac{d R}{d \epsilon} + \frac{d R}{d \pi}\right) n\, d t - \frac{a\,.(1-e^2)}{e} \,.\, \frac{d R}{d \epsilon} n\, d t$$

$$= \frac{a}{e} \sqrt{(1-e^2)}\, [1 - \sqrt{(1-e^2)}] \frac{d R}{d \epsilon} n\, d t + \frac{a}{e} \sqrt{(1-e^2)} \frac{d R}{d \pi} \,.\, n\, d t.$$

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* $\frac{d R}{n\, d t}$, $\frac{d R}{d \epsilon}$, are symbols of like signification, and, after the establishment of the formulæ of the differential calculus, of plain signification. Short processes of demonstration are frequently little else than compendious expressions: these latter must be posterior to methods and formulæ; and the enriching the language of analysis, although the secondary or collateral object of the differential calculus, has proved one of its greatest benefits. When its formulæ are established, the principles on which they were established may be put aside from consideration; there is, for instance, no notion of variability to be attached to such symbols as

$$\frac{d R}{d \pi}, \ \frac{d R}{d \epsilon}, \ \text{\&c.}$$

† Or thus, $\frac{d R}{n\, d t} = \frac{d R}{d v} \,.\, \frac{d v}{n dt}$,

$$\frac{d R}{d \epsilon} = \frac{d R}{d v} \,.\, \frac{d v}{d \epsilon}, \ \text{but} \ \frac{d v}{n d t} = \frac{d v}{d \epsilon}; \ \therefore \ \frac{d R}{n\, d t} = \frac{d R}{d \epsilon}.$$

But, if the preceding expression be not more commodious for computation, there is this thing remarkable in it, namely, that the variation of the element e is expressed by the *partial differential coefficients* (*partial differences* the French call them) of the same quantity R computed for the elements; and, moreover, the coefficients of these partial differential coefficients, are functions of the elements themselves. This latter circumstance indeed belongs to the former expression for δe, but neither that nor the previous one are attached to the variation of π. We must farther consider this point.

$\delta a = -2a^2 . dR$: and since, as we have seen (pp. 388, 389.)

$$dR = \frac{dR}{n\,dt}\,n\,dt = \frac{dR}{d\epsilon}\,n\,dt,$$

the expression is under the same peculiarity of conditions as δe is. A question therefore naturally suggests itself, whether $\delta \pi$ is excluded by its composition, or the law of its formation, from like conditions, or whether, by virtue of certain transformations, it may not be made to participate in them.

If R be considered to be a function of r and v, that is, if (see p. 273.)

$$R = \frac{m' r}{r'^2} \cos.\, (v - v') - \frac{m'}{\sqrt{[r^2 - 2\,r\,r' \cos.\,(v - v') + r'^2]}},$$

then the partial differential coefficient $\frac{dR}{de}$ can only proceed from R containing r, the value of which is

$$r = \frac{a.(1 - e^2)}{1 + e\,.\cos.\,(v - \pi)}.$$

Hence,

$$\frac{dR}{de} = \frac{dR}{dr} \cdot \frac{dr}{de}$$

$$= \left(dR\,\frac{1}{dr} - \frac{dR}{dv} \cdot \frac{dv}{dr}\right) \frac{dr}{de}.$$

If we substitute for $\frac{dv}{dr}$, $\frac{dr}{de}$ their values to be derived from the preceding expression for r, we shall have (see p. 386.)

$$\frac{2e + (1 + e^2)\cos.(v - \pi)}{he^2 \sin.(v - \pi)} \cdot \frac{dR}{dv} dt = \frac{h}{ae} \cdot \frac{dR}{de} dt +$$

$$\frac{2e + (1 + e^2)\cos.(v - \pi)}{e^2.\sin.(v - \pi)(1 + e.\cos.v - \pi)^2} h^2.dR,$$

and consequently, instead of the former value of $\delta\pi$ (see p. 386. l. 13.) we shall have this

$$\delta\pi = -\frac{h}{ae} \cdot \frac{dR}{de} dt - \frac{h^2}{e} \cdot \frac{\sin.(v - \pi)[2 + e\cos.(v - \pi)]}{[1 + e\cos.(v - \pi)]^2} dR.$$

Here $\delta\pi$ is under one of the predicaments that δa, δe are, namely, that of being expressed partly by the aid of $\frac{dR}{de}$ and of $\frac{dR}{d\epsilon}$ $\left(\text{since } dR = \frac{dR}{d\epsilon} n\,dt\right)$; but the coefficient of $\frac{dR}{d\epsilon}$, instead of being a function of the elements, is a function of v. In this respect, therefore, its expression differs from those of δa and δe.

We must observe, however, that the expressions for δa, δe, are true, when R is a function of nt and of the arbitrary quantities a, e, &c.: whereas, as it is plain from the mode of deducing the expression, $\frac{dR}{de}$ is the partial differential coefficient, on the supposition that e enters into R solely from being contained in r. Now, (see p. 388.) the expression for the true anomaly (v) in terms of the mean (nt) is

$$v = nt + \epsilon + 2e.\sin.(nt + \epsilon - \pi) + \&c.$$

this conversion, then, of the true into the mean anomaly, introduces e: consequently, the preceding expression $\frac{dR}{de}$, supposing R to be a function of nt, &c. is an imperfect value of that differential coefficient. We must complete its value, then, and on the following grounds; namely, that R, a function of v and e, is to be converted into R a function of $n\,t$ and e: therefore,

$$\frac{d\,R}{d\,v}.d\,v + \left(\frac{d\,R}{d\,e}\right)d\,e = \frac{d\,R}{ndt}.n\,d\,t + \frac{d\,R}{d\,e}.d\,e,$$

in which $\left(\frac{d\,R}{d\,e}\right)$ is used to denote the *imperfect* value of the partial differential coefficient.

We must now find $d\,v$ in terms of the differentials of $n\,t$ and of e, and this is most easily done by means of the eccentric anomaly (u): thus, (see p. 31.)

$$n\,t = u - e\sin.\,u;$$

$$\therefore n\,d\,t = du\,(1 - e\cos.\,u) - d\,e.\sin.\,u,$$

$$\text{but } \cos.\,u = -\frac{e + \cos.\,(v-\pi)}{1 + e\cos.\,(v-\pi)};$$

$$\therefore d\,u = \frac{d\,v\,(1-e^2) - d\,e.\sin.\,(v-\pi)}{\sqrt{(1-e^2)}\,[1 + e\cos.\,(v-\pi)]};$$

consequently,

$$ndt = \frac{(1-e^2)^{\frac{3}{2}}\,d\,v}{[1+e\cos.(v-\pi)]^2} - \frac{\sqrt{(1-e^2)}.\sin.\,(v-\pi)}{[1+e.\cos.\,(v-\pi)]^2}[2+e\cos.\,(v-\pi)]\,d\,e.$$

Substitute this in the preceding equation, and equate the coefficients of like terms, and there will result

$$\left(\frac{d\,R}{d\,e}\right) = \frac{d\,R}{d\,e} - \frac{\sqrt{(1-e^2)}.\sin.\,(v-\pi)}{[1+e\cos.\,(v-\pi)]^2}.[2 + e\cos.\,(v-\pi)].\frac{d\,R}{n\,d\,t}.$$

Now, $\left(\frac{d\,R}{d\,e}\right)$ is the partial differential coefficient used in expressing the former value of $\delta\pi$ (see p. 391.): substitute therefore in that value the expression just obtained for $\left(\frac{d\,R}{d\,e}\right)$: and, since,

$$\frac{h}{ae}\sqrt{(1-e^2)}\,\frac{dR}{n\,dt}.\,dt = \frac{1-e^2}{\sqrt{a}\,.e}\,dR \times \frac{1}{n}$$

$$= \frac{a(1-e^2)}{e}\,dR$$

$$= \frac{h^2}{e}\,dR,$$

it is plain that $\delta\,\pi$ will be reduced to this value

$$\delta\,\pi = -\frac{h}{ae}\,.\,\frac{dR}{de}.\,dt$$

$$= -\frac{a.\sqrt{(1-e^2)}}{e}\,.\,\frac{dR}{de}.\,n\,dt,$$

an expression, considering the intricacy of the subject of computation, of remarkable simplicity.

The variations of three of the elements, then, are now expressed in terms of such partial differential coefficients of R, as $\frac{dR}{d\epsilon}$, $\frac{dR}{d\pi}$, $\frac{dR}{de}$; and, moreover, the coefficients of the terms are functions of the elements themselves. In these cases, R is a function of the mean anomaly and of the elements of the orbit: of nt and of a, e, π, &c. When R is a function of ρ, v, and s, there are only five arbitrary quantitities, viz. a, e, π, γ and θ: but the conversion of v into a function of nt, since (see pp. 40, &c.) it is effected subsequently to an integration, introduces, by virtue of that integration, an additional arbitrary quantity ϵ, the epoch of the mean longitude, or the time at which the planet is in the perihelion of the orbit. There will then be six arbitrary quantities; but, if we abstract the inclination, and consequently the longitude of the nodes, R will be a function only of nt and of a, e, π, ϵ, and accordingly we shall have this equation of condition for the variations of the elements,

$$\frac{dR}{da}\,\delta\,a + \frac{dR}{de}\,\delta\,e + \frac{dR}{d\pi}\,\delta\,\pi + \frac{dR}{d\epsilon}\,\delta\,\epsilon = 0,$$

or, substituting for the values of $\delta\,a$, $\delta\,e$, $\delta\,\pi$,

$$\left\{\begin{array}{l} \qquad\frac{dR}{da}\times -2a^2.dR \\ +\frac{dR}{de}\left(\frac{a}{e}\sqrt{(1-e^2)}[1-\sqrt{(1-e^2)}]dR+\frac{a\sqrt{(1-e^2)}}{e}.\frac{dR}{d\pi}n\,dt\right) \\ \qquad\qquad\qquad + \frac{dR}{d\pi}\times -\frac{a\sqrt{(1-e^2)}}{e}.\frac{dR}{de}n\,dt \end{array}\right\} =$$

$$-\frac{dR}{d\epsilon}.\delta\epsilon = \text{(see p. 389.)} - \frac{dR}{n\,dt}\delta\epsilon.$$

Hence, (since the second part of the second term destroys the third term), we have, on dividing by $-\frac{dR}{n\,dt}$,

$$\delta\epsilon = 2a^2.\frac{dR}{da}n\,dt - \frac{a\sqrt{(1-e^2)}}{e}[1-\sqrt{(1-e^2)}]\frac{dR}{de}n\,dt.$$

So that this variation also, with regard to its expression, is under the same predicaments as the three preceding variations.

We must now restore what, for convenience of computation, we have abstracted; namely, the inclination of the plane of the orbit, and consider its variation and that of the longitude of the nodes.

Now the third equation, (see p. 383.) is

$$d^2(\rho s) + \left(\frac{s}{\rho^2.(1+s^2)^{\frac{3}{2}}} + \frac{1}{\rho}.\frac{dR}{ds}\right)dt^2 = 0 \ldots\ldots\ldots (3).$$

When the disturbing force acts, as in the former case (see p. 383.)

$$\begin{aligned} d^2.\rho s &= d.d\rho s = d(s\,d\rho + s\,\delta\rho + \rho\,ds + \rho\,\delta s) \\ [D] \qquad &= s\,d^2\rho + 2\,ds\,d\rho + s\,d\delta\rho + \rho\,d^2 s + \rho\,d\delta s \\ &\quad + ds\,\delta\rho + d\rho\,\delta s. \end{aligned}$$

But, $\delta\rho = 0$, $\delta s = 0$, and, besides, we have the elliptical equation,

$$d^2(\rho s) + \frac{s}{\rho^2.(1+s^2)^{\frac{3}{2}}}dt^2 = 0,$$

$$\text{or, } s\,d^2\rho + 2\,ds.d\rho + \rho\,d^2 s + \frac{s}{\rho^2.(1+s^2)^{\frac{3}{2}}}.dt^2 = 0.$$

If, therefore, the preceding value (D) of $d^2.\rho s$ be substituted in the equation (3), and, then, that equation be reduced by the means just pointed out, there will result

$$s\,d\,\delta\,\rho + \rho\,d\,\delta\,\rho + \frac{1}{\rho}\cdot\frac{d\,R}{d\,s}\,d\,t^2 = 0,$$

but, (see p. 383.) $d\,\delta\,\rho = -\dfrac{d\,R}{d\,\rho}d\,t^2 + \dfrac{s}{\rho}\cdot\dfrac{d\,R}{d\,s}$;

$$\therefore\ d\,\delta\,s = \left(\frac{s}{\rho}\cdot\frac{d\,R}{d\,\rho} - \frac{1+s^2}{\rho^2}\cdot\frac{d\,R}{d\,s}\right)d\,t^2.$$

We must now apply this expression by taking the variation of some equation similar to the one of p. 384, and which involves one or more arbitrary quantities. Such an equation we may immediately find in this finite value of s (see p. 38.)

$$s = \gamma\,.\sin.\,(v - \theta),$$

($\gamma = \tan.\,\phi$, ϕ being the inclination);

$$\therefore\ d\,s = d\,v\,.\gamma\,.\cos.\,(v - \theta),$$

and

$$d\,\delta\,s = d\,\delta\,v\,.\gamma\cos.\,(v-\theta) + d\,v\cos.\,(v-\theta)\,\delta\,\gamma + d\,v\,\gamma\sin.\,(v - \theta)\,\delta\,\theta,$$

$$\text{or, } \left(\frac{s}{\rho}\cdot\frac{d\,R}{d\,\rho} - \frac{(1+s^2)}{\rho^2}\cdot\frac{d\,R}{d\,s}\right)d\,t^2 =$$

$$-\gamma.\cos.\,(v-\theta)\,.\frac{1}{\rho^2}\cdot\frac{dR}{d\,v}dt^2 + dv.\,\delta\gamma.\cos.\,(v-\theta) + dv\delta\theta.\gamma\sin.\,(v-\theta),$$

the equation of p. 384. contained the variation of only one arbitrary quantity (a), but this contains the variations of two (γ and θ): we must employ, therefore, for the purpose of elimination, the equation of condition,

$$\delta\,s = 0,$$

$$\text{or, }\quad \delta\,\gamma.\sin.\,(v - \theta) - \delta\,\theta\,.\gamma\cos.\,(v - \theta) = 0,$$

and, by eliminating, there results,

$$d\,v\,.\,\delta\,\gamma = \left(\frac{s}{\rho}\cdot\frac{d\,R}{d\,\rho} + \frac{\gamma}{\rho^2}\cos.\,(v-\theta)\frac{d\,R}{d\,v}\right)\cos.\,(v - \theta)\,.\,d\,t^2$$

$$-\frac{1+s^2}{\rho^2}\,.\cos.(v-\theta)\,\frac{d\,R}{d\,s}\,d\,t^2,$$

which may be differently expressed : for, since $h\,d\,t=\rho^2\,d\,v$, and and $s=\gamma\,.\sin.(v-\theta)$, there will result, these being substituted,

$$\delta\gamma=\frac{\gamma}{h}\left(\rho\,.\sin.(v-\theta)\,.\frac{d\,R}{d\,\rho}+\cos.(v-\theta)\,.\frac{d\,R}{d\,v}\right)\cos.(v-\theta)\,d\,t$$

$$-\frac{(1+s^2)}{h}\cos.(v-\theta).\frac{d\,R}{d\,s}\,d\,t,$$

or,

$$\delta\phi\times\cos.^2\phi\,(=\delta\gamma)=$$

$$\frac{\gamma\,\rho^2}{h^2}\left(\rho\,.\sin.(v-\theta)\,\frac{d\,R}{d\,\rho}+\cos.(v-\theta)\frac{d\,R}{d\,v}\right)\cos.(v-\theta)\,d\,v$$

$$-\,(1+s^2)\,\frac{\rho^2}{h^2}\cos.(v-\theta)\,.\frac{d\,R}{d\,s}\,d\,v.$$

The expression for the variation of the longitude of the nodes is, by means of the equation of condition, (p. 395. l. 21.) reduced to this

$$\delta\theta=\frac{1}{h}\left(\rho\,.\sin.(v-\theta)\,\frac{d\,R}{d\,\rho}+\cos.(v-\theta).\frac{d\,R}{d\,v}\right)\sin.(v-\theta).d\,t$$

$$-\frac{(1+s^2)}{h\,\gamma}\sin.(v-\theta)\,\frac{d\,R}{d\,s}\,.\,d\,t,$$

$$\text{or, }=\frac{\rho^2}{h^2}\left(\rho\,.\sin.(v-\theta)\,.\frac{d\,R}{d\,\rho}+\cos.(v-\theta)\,\frac{d\,R}{d\,v}\right)\sin.(v-\theta).\,d\,v$$

$$-\,(1+s^2)\frac{\rho^2}{h^2\,\gamma}\,\sin.(v-\theta)\,\frac{d\,R}{d\,s}\,d\,v.$$

From these expressions the regression of the nodes and the change or variation of the inclination of the plane of the disturbed body's orbit may be computed ; and very expeditiously by the second and fourth expression, if R should be expressed by a function of ρ, v and s. But if R should be expressed, as it is in the case of the planets, by a function of $n\,t$, a, and other quantities, then the variations of the inclination and of the place of the node could not be computed immediately from the preceding expressions. A previous resolution of sin. $(v-\theta)$, cos. $(v-\theta)$, &c. into sines and cosines of arcs composed of nt and other quantities would be necessary. This relates to the mere matter and convenience of computation. With regard to the analytical mode of expressing the variations of the node and inclination, the above formulæ want the characteristics, or the predicaments that the formulæ for the variations of the other elements possess (see p. 393. ll. 9, 10, &c.). It is the object of the succeeding process to shew that they may be invested with them.

Since R is a function of ρ, s, v, and ρ is a function of s, and s of θ, we have,

$$\frac{dR}{d\theta}=\frac{dR}{d\rho}\cdot\frac{d\rho}{ds}\cdot\frac{ds}{d\theta}+\frac{dR}{ds}\cdot\frac{ds}{d\theta},$$

and in order to obtain $\frac{d\rho}{ds}$, $\frac{ds}{d\theta}$, we have these finite equations,

$$\rho=\frac{h^2(1+\gamma^2)}{\sqrt{(1+s^2)}+e.\cos.(v-\pi)},$$

$$s=\gamma.\sin.(v-\theta).$$

Hence, very nearly,

$$\frac{d\rho}{ds}\cdot\frac{ds}{d\theta}=\frac{\gamma^2\rho^2.\cos.(v-\theta).\sin.(v-\theta)}{h^2},$$

and

$$\frac{\gamma\rho.\sin.(v-\theta)\cos.(v-\theta)}{h}\cdot\frac{dR}{d\rho}-\frac{h}{\rho}\cos.(v-\theta)\frac{dR}{ds}=\frac{h}{\gamma\rho}\cdot\frac{dR}{d\theta}.$$

or, assuming $\frac{1}{h}$ as the approximate value of $\frac{h}{\rho}$,

$$\frac{\gamma\rho}{h}\sin.(v-\theta).\cos.(v-\theta)\frac{dR}{d\rho}-\frac{1}{h}\cos.(v-\theta).\frac{dR}{ds}=\frac{1}{h\gamma}\cdot\frac{dR}{d\theta}.$$

Hence, if we reject in the expression for $\delta\gamma$ the term that involves s^2, we shall have

$$\delta\gamma=\frac{1}{h\gamma}\cdot\frac{dR}{d\theta}dt+\frac{\gamma}{h}\cos.^2(v-\theta)\frac{dR}{dv}dt.$$

This expression is only partially under the predicaments of the preceding ones of p. 393. $\frac{dR}{d\theta}$ is indeed the partial differential of R taken relatively to the element θ, but the last term $\left(\frac{\gamma}{h}\cos.^2(v-\theta)\frac{dR}{dv}\right)$ involves $\frac{dR}{dv}$, and also the coefficient $\cos.^2(v-\theta)$ which is not a function of the elements. And, since v is the longitude measured, not on the plane of the body's orbit, but on a plane such as that of the ecliptic, $\frac{dR}{dv}$ is not equal (see pp. 388, 389.) to $\frac{dR}{d\pi}+\frac{dR}{d\epsilon}$; therefore, in two points, the above expression for $\delta\gamma$ is dissimilar to the expressions for the other variations. The dissimilarity, however, may be made to disappear by measuring the longitudes of the body's place, and of the node, on the plane of the body's orbit; which amounts, analytically, to the transformation of R, a function of v, θ, &c. into a function of $v_{\prime}$, θ, &c. supposing $v_{\prime}$, to be the longitude on the body's orbit, and θ to be that longitude measured on the same orbit *.

In order to effect this transformation, we have by the property of spherical triangles (see *Trig.* p. 136.)

$$\tan.(v-\theta)=\cos.\phi.\tan.(v_{\prime}-\theta).\ \ldots\ldots\ldots(1),$$

and by the properties of analytical functions †

$$\left(\frac{dR}{d\theta}\right)d\theta+\frac{dR}{dv}dv=\frac{dR}{d\theta}d\theta+\frac{dR}{dv_{\prime}}dv_{\prime}\ldots\ldots\ldots(2),$$

in which $\left(\frac{dR}{d\theta}\right)$ is used to denote the partial differential coefficient of R when R is a function of v, θ, &c. and to distinguish it from $\frac{dR}{d\theta}$ in which R is supposed to be a function of $v_{\prime}$, θ, &c.

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* θ, the real longitude of the node when measured from the intersection of the plane of the ecliptic (if that be the fixed plane) and of the orbit of the planet, is the longitude of the node on the planet's orbit (see *Astronomy*, p. 254.).

† *Principles of Analytical Calculation*, pp. 79, &c. &c.

If we take the differential of the first equation (1), and suppose ϕ not to vary,

$$d v_{,} = \sec.\phi.\frac{\cos.^2(v_{,}-\theta)}{\cos.^2(v-\theta)}.d v + \left(1-\frac{\sec.\phi.\cos.^2(v_{,}-\theta)}{\cos.^2(v-\theta)}\right) d\theta,$$

which value being substituted in the second equation (2), and the coefficients of the terms affected with $d\theta$ equated, there results,

$$\left(\frac{dR}{d\theta}\right) = \frac{dR}{d\theta} + \frac{dR}{dv}\left(\frac{\cos.\phi.\cos.^2(v-\theta)}{\cos.^2(v_{,}-\theta)} - 1\right).$$

But, from equation (1),

$$\frac{1}{\cos.^2(v_{,}-\theta)} = 1. + \tan.^2(v-\theta).\sec.^2\phi;$$

$$\therefore \left(\frac{dR}{d\theta}\right) = \frac{dR}{d\theta} + \frac{dR}{dv}\left[\cos.\phi\cos.^2(v-\theta) + \sec.\phi\sin.^2(v-\theta) - 1\right]$$

$$= \frac{dR}{d\theta} + \frac{dR}{dv}\left(\frac{1-\cos.\phi}{\cos.\phi} - \frac{\sin.^2\phi}{\cos.\phi}\cos.^2(v-\theta)\right)$$

$$= \frac{dR}{d\theta} + \frac{2\sin.^2\frac{\phi}{2}}{\cos.\phi}.\frac{dR}{dv} - \tan.\phi\left[\sin.\phi\cos.^2(v-\theta)\right]\frac{dR}{dv}.$$

Hence, (see p. 397. line the last.)

$$\delta\gamma = \frac{1}{h\gamma}.\frac{dR}{d\theta}dt + \frac{2\sin.^2\frac{\phi}{2}}{h\sin.\phi}.\frac{dR}{dv}dt, \text{ very nearly,}$$

since, $\left(\gamma \text{ being } = \tan.\phi = \frac{\sin.\phi}{\cos.\phi}\right)$,

$$\frac{\gamma}{h}\cos.^2(v-\theta)\frac{dR}{dv} - \frac{\sin.\phi}{h}\cos.^2(v-\theta)\frac{dR}{dv}$$

$$= \frac{1}{h}\tan.\phi.2\sin.^2\frac{\phi}{2}\cos.^2(v-\theta)\frac{dR}{dv},$$

a term much less than the other terms in the expression for $d\gamma$, when ϕ, the inclination of the planes, is a small quantity.

By equating the terms affected with dv,

$$\frac{dR}{dv}\cdot\frac{\cos.\phi\cos^2.(v-\theta)}{\cos.^2(v_,-\theta)}=\frac{dR}{dv_,},$$

$$\text{or, }\frac{dR}{dv}\left(\frac{1-\sin.^2\phi\cos.^2(v-\theta)}{\cos.\phi}\right)=\frac{dR}{dv_,};$$

$$\therefore\ \frac{2\sin.^2\frac{\phi}{2}}{h\sin.\phi}\cdot\frac{dR}{dv}=\frac{2\sin.^2\frac{\phi}{2}}{h\tan.\phi}\,\frac{dR}{dv_,},\text{ very nearly,}$$

$$\text{and }\ \delta\gamma=\frac{dt}{h\tan.\phi}\cdot\left(\frac{dR}{d\theta}+2\sin^2.\frac{\phi}{2}.\frac{dR}{dv_,}\right).$$

$v_,$ is the body's longitude measured on the plane of the body's orbit; therefore (see pp. 388, 389.)

$$\frac{dR}{dv_,}=\frac{dR}{d\epsilon}+\frac{dR}{d\pi};$$

consequently,

$$\delta\gamma=\frac{dt}{h\gamma}\left[\frac{dR}{d\theta}+2\sin^2.\frac{\phi}{2}\left(\frac{dR}{d\epsilon}+\frac{dR}{d\pi}\right)\right]\ *,$$

and (see p. 393.) the variation $\delta\gamma$ is now expressed by a formula possessing all the characteristics of the former variations.

In this and in the former expression for $\delta\gamma$ (see p. 399.) the partial differential coefficient $\frac{dR}{d\theta}$ enters: but $\frac{dR}{d\theta}$, in the two cases, is of different values, since R in the first case is a function of v, θ, ϕ, &c. and in the second of v, θ, ϕ, &c.

The expression may still farther be varied, by transforming R into a function of new quantities, the relations between these latter and the quantities v, θ, &c. being determined by certain equations: for instance, if instead of $v_,$ we assume a quantity u and determine the relation between $v_,$ and u by this equation,

$$du=dv_,-(1-\cos.\phi)\,d\theta;$$

then, since R, instead of being a function of v, θ, &c. is now to become a function of u, θ, &c.

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* This is the same expression as that which Laplace, by a different method, has deduced in the supplement au X. Liv. *de Mec. Cel.*

we have

$$\frac{dR}{dv_{,}}dv_{,} + \left(\frac{dR}{d\theta}\right)d\theta = \frac{dR}{du}du + \frac{dR}{d\theta}d\theta$$

$$= \frac{dR}{du}[dv_{,} - (1 - \cos.\phi)\,d\theta] + \frac{dR}{d\theta}d\theta;$$

therefore, by a comparison like the preceding one of p. 399.

$$\left(\frac{dR}{d\theta}\right) = \frac{dR}{d\theta} - 2\sin.^2\frac{\phi}{2}.\frac{dR}{dv_{,}};$$

consequently, (see p. 400. l. 4.)

$$\delta\gamma = \frac{dt}{h\gamma}.\frac{dR}{d\theta}\,{*}.$$

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[*] This certainly is a very simple expression, and it is the same which Lagrange, *Mem. Inst.* 1808. pp. 62, 64, &c. and Poisson, *Ecole Polytechnique,* tom IX. have, by methods differing both from the preceding and from each other, deduced. The simplification, however, obtained by the last step is more apparent than real, since it is obtained by introducing a quantity u the relation of which to v, &c. is determined only by a differential equation.

A simpler form may, in like manner, be given to some of the other variations, by transforming R into a function of different quantities: thus, we have, very nearly,

$$d\epsilon = -\frac{an\sqrt{(1-e^2)}}{e}dt\left(\frac{dR}{de} - \frac{2ae}{\sqrt{(1-e^2)}}.\frac{dR}{da}\right).$$

Assume $dq = da + \frac{2ae}{\sqrt{(1-e^2)}}de;$

then, as before (p. 401, ll. 2, 3.)

$$\left(\frac{dR}{de}\right)de + \frac{dR}{da}da = \frac{dR}{de}de + \frac{dR}{dq}dq.$$

Substitute for dq its assumed value and equate the coefficients of like terms, and then

$$\frac{dR}{dq} = \frac{dR}{da},$$

$$\left(\frac{dR}{de}\right) = \frac{dR}{de} + \frac{dR}{dq}.\frac{2ae}{\sqrt{(1-e^2)}};$$

∴

The only variation that remains to be invested with the properties which the other variations possess, is $\delta\theta$: now, since

$$\frac{dR}{d\theta} = \left(\frac{dR}{d\rho}\cdot\frac{d\rho}{ds} + \frac{dR}{ds}\right)\frac{ds}{d\theta},$$

$$\text{and } \frac{dR}{d\gamma} = \left(\frac{dR}{d\rho}\cdot\frac{d\rho}{ds} + \frac{dR}{ds}\right)\frac{ds}{d\gamma},$$

$$\frac{dR}{d\theta}\cdot\frac{ds}{d\gamma} = \frac{dR}{d\gamma}\cdot\frac{ds}{d\theta}.$$

But (see p. 395.) the equation of condition is

$$\frac{ds}{d\gamma}\,\delta\gamma + \frac{ds}{d\theta}\,\delta\theta = 0.$$

Substitute for the value of $\delta\gamma$ in p. 397, and

$$\delta\theta = -\frac{1}{h\gamma}\cdot\frac{dR}{d\theta}\cdot\frac{\frac{dR}{d\gamma}}{\frac{dR}{d\theta}} - \frac{\gamma}{h}\cos.^2(v-\theta)\cdot\frac{dR}{dv} \times \frac{\frac{ds}{d\gamma}}{\frac{ds}{d\theta}};$$

$$\text{but } \frac{ds}{d\gamma} = \sin.\,(v-\theta),$$

$$\text{and } \frac{ds}{d\theta} = -\gamma\cos.\,(v-\theta);$$

$$\therefore \delta\theta = -\frac{1}{h\gamma}\cdot\frac{dR}{d\gamma} + \frac{1}{h}\cos.\,(v-\theta)\sin.\,(v-\theta)\,\frac{dR}{dv}.$$

In order to get rid of $\frac{dR}{dv}$ and its coefficient, we must use a transformation similar to the preceding one of p. 398. In the

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$$\therefore \left(\frac{dR}{de}\right) - \frac{dR}{da}\cdot\frac{2ae}{\surd(1-e^2)} = \frac{dR}{d\epsilon};$$

consequently,

$$d\epsilon = -\frac{an\surd(1-e^2)}{e}\cdot dt\cdot\frac{dR}{d\epsilon}.$$

present transformation, however, since a different expression for $\delta\theta$ is required, θ must be supposed constant, and γ, v, and $v_{,}$ to vary in the equation,

$$\tan.\,(v_{,}-\theta)\cos.\,\phi = \tan.\,(v-\theta),$$

in which supposition,

$$dv_{,} = \delta\phi\,.\cos.^2\phi\,\tan.\,(v_{,}-\theta)\cos^2.\,(v_{,}-\theta) + \frac{dv\cos.^2(v_{,}-\theta)}{\cos.\,\phi\,.\cos^2.\,(v-\theta)},$$

which value being substituted in

$$\left(\frac{dR}{d\gamma}\right)\delta\gamma + \frac{dR}{dv}\delta v = \frac{dR}{d\gamma}\delta\gamma + \frac{dR}{dv_{,}}\delta v_{,}{}^{*},$$

and, the coefficients being equated,

$$\left(\frac{dR}{d\gamma}\right) = \frac{dR}{d\gamma} + \frac{dR}{dv}\gamma\,.\cos.^3\phi\,.\cos^2.\,(v-\theta)\tan.\,(v_{,}-\theta)$$

$$= \frac{dR}{d\gamma} + \frac{dR}{dv}\gamma\,.\cos.^2(v-\theta)\tan.\,(v_{,}-\theta)\,(1-\sin^2.\,\phi)$$

$=$ nearly, (since $\gamma\,.\sin^2.\,\phi$ is a very small quantity),

$$\frac{dR}{d\gamma} + \gamma\,.\cos.\,(v-\theta)\sin.\,(v-\theta)\,.\frac{dR}{dv}.$$

Substitute this value for $\frac{dR}{d\gamma}$ in the expression of p. 402. l. 12. and there results,

$$\delta\theta = -\frac{1}{h\gamma}\,.\frac{dR}{d\gamma}.$$

The variations of the six elements may now be exhibited under one view,

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* Since ϕ is very small, $d\gamma = d(\tan.\,\phi)$ is written instead of $d\phi$. $\left(\frac{dR}{d\gamma}\right)$ is here used, after a manner similar to that in p. 398, and to distinguish it from $\frac{dR}{d\gamma}$, the partial differential coefficient of R, when R is transformed into a function of new quantities.

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$$\delta a = -2a^2.dR, \text{ or } = -2a^2.\frac{dR}{d\epsilon}n\,dt \ldots\ldots\ldots\ldots 385, 389.$$

$$\delta e = \frac{a}{e}\sqrt{(1-e^2)}[1-\sqrt{(1-e^2)}]\frac{dR}{d\epsilon}n\,dt + \frac{a}{e}\sqrt{(1-e^2)}\frac{dR}{d\pi}n\,dt \ldots\ldots 389.$$

$$\delta\pi = -\frac{a\sqrt{(1-e^2)}}{e}.\frac{dR}{de}n\,dt \ldots\ldots\ldots\ldots\ldots\ldots 393.$$

$$\delta\epsilon = 2a^2.\frac{dR}{da}n\,dt - \frac{a\sqrt{(1-e^2)}}{e}[1-\sqrt{(1-e)}]\frac{dR}{de}n\,dt \ldots\ldots 394.$$

$$* \delta\gamma = \frac{1}{h\gamma}.\frac{dR}{d\theta}.dt \ldots\ldots\ldots\ldots\ldots\ldots 401.$$

$$\text{or,} = \frac{1}{h\gamma}\left[\frac{dR}{d\theta} + 2\sin^2.\frac{\phi}{2}\left(\frac{dR}{d\epsilon} + \frac{dR}{d\pi}\right)\right]dt \ldots\ldots\ldots 400.$$

$$\delta\theta = -\frac{1}{h\gamma}.\frac{dR}{d\gamma}dt \ldots\ldots\ldots\ldots\ldots\ldots 403.$$

The preceding expressions depend, and mainly, on the partial differentials of the quantity R. In that respect, therefore, they are similar; they are also, the formula for R being supposed to be established, of easy application.

The formula for R, which is requisite for such application, will be similar to that which was given in p. 279: but it must be extended so as to comprehend the terms that will arise on introducing the condition of the plane's inclination.

But the formulæ of the variations, although they possess a sort of symbolical similarity and are convenient for arithmetical computation, are derived by no simple processes. The fault, if it may be so called, does not lie entirely with the processes. The objects of investigation are abstruse. There is no immediate nor short connexion between the principle and Law of Gravity, and that, for instance, its result, which is called the Progression of the Perihelion.

It is also, in part at least, to be ascribed to the *nature* of the research, that its processes are not obvious, but rather indirect. What is the obvious and direct process which the consideration of the *progression* of the perihelion or of the variation of the axis-major suggests? The obvious and direct processes belong to much more simple subjects of enquiry; to the regression of the nodes, for instance, which is indeed one of the variations of the elements, and, therefore, an exception to what was said at p. 404. ll. 19, 20. But let any one, versed in these studies, and taking for his model the 30th Proposition of the third Book, or any other of the Propositions of the *Principia*, attempt similar constructions for the variations of the position and of the magnitude of the axis-major, and he will soon experience the very unaccommodating nature of these latter subjects. They are much too stubborn or subtle to yield to the ordinary modes of attack.

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* It must be recollected (see pp. 400, 401.) that R in these two expressions is not a function of the same quantities.

The elements of a planet's orbit being considered as certain arbitrary quantities introduced by the integration of the differential equations, the research of the variations of the elements immediately becomes a merely mathematical enquiry. The object of research, so far from directing the research, is altogether lost sight of. We come upon it all at once when the investigation is finished. If this be an imperfection in science it does not belong to the infancy of science.

As science advances, its processes become more compendious, and have less *affinity*, if we may so express ourselves, with their subjects.

The variation of the axis-major, which, by direct and obvious methods, is most difficult of approach, is, by the preceding methods, most easily investigated. Analogy would lead us to a different conclusion.

Newton, as it has been observed, did not treat of the variations of all the elements, nor did those mathematicians who formed, what may be called, the first *set* of his successors, and who endeavoured to establish the system of gravitation more firmly than its great founder had time to do. Clairaut, Dalembert, and Thomas Simpson determined the progression of the Lunar Apogee; the first by the method described in Chapters IX, XIII: the two latter by the method of indeterminate coefficients (see pp. 209, &c.) In determining the regression of the nodes and the variation of the inclination of the plane, they adopted, in fact, Newton's method: than which, as an independent and simple method nothing can be better. The first step, indeed, towards the *variation of the parameters* (which is now the method) was made by a contemporary of the above-mentioned mathematicians.

Euler *, in his *Theoria Lunæ*, p. 8, resolved the third differential equation into two others; one expounding the regression of the node, and the other the connected change in the plane's inclination. The invention, however, of the general formulæ belongs to the mathematicians of the second *set*, of which the most distinguished are Lagrange and Laplace. The former of these mathematicians deduced, in the *Berlin Memoirs* for 1774, that remarkable formula (see pp. 327, 380, 385.) from which it follows that the mean distances of the planets are subject to no *secular* variation from disturbing forces. Laplace arrived at this latter result by a different process: and the subject continued to be cultivated till Lagrange was enabled to express the variations of the elements by the formulæ of p. 404. This he did in the *Memoirs of the Institute* for 1809, by processes strictly analytical, but very long, and on principles not naturally, we may say, suggested by the subject of enquiry. Laplace in the supplement to the 10th Volume of the *Mec. Cel.* by more simple means, converted the formulæ he had already invented, into Lagrange's; and, in a Memoir inserted in the 9th Volume of the *Ecole Polytechnique*, M. Poisson has, and by a peculiar method, arrived at the same formulæ.

In the next Chapter we will deduce certain simple results from the preceding formulæ and illustrate the formulæ by examples. For that purpose it will be requisite to extend (see p. 279.) the expression for R in order to adapt it to their application. That, therefore, will be the first operation.

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* 'Mais Euler est le premier qui ait cherché a les determiner par l'analyse, ses formules etant de peu d'usage par leur complication', &c. Lagrange, *Mem. Inst.* 1809. p. 364.

CHAP. XXII.

Deduction of the constant Parts of the Development of R. *Expressions for the Secular Variations of the Elements. Variations of the Eccentricities of the Orbits of Jupiter and Saturn. Theorem for shewing that their Eccentricities can neither increase nor decrease beyond certain Limits. Diminution of the Eccentricity of the Earth's Orbit. It is the Cause of the Acceleration of the Moon's Mean Motion. Its Value computed from the disturbing Forces of the Planets. Thence, the Secular Equation of the Moon's Acceleration computed. Variation of the Longitude of the Perihelion: sometimes a Progression, at other times a Regression. The Progressions of the Perihelia of Jupiter and Saturn computed. Variations of Inclination and of Node. Theorem for shewing that the Inclinations of the Planes of Orbits oscillate about a mean Inclination. The Mean Motions of Nodes, with reference to the Ecliptic, sometimes Progressive, at other times Regressive: but, with reference to the Orbit of the disturbing Planet, always Regressive. The Moon's Nodes. The Quantity of their Regression computed. Variation of the Obliquity of the Ecliptic: Progression of the Equinoxes; both caused by the disturbing Forces of the Planets: their Quantities computed. The Length of the Tropical Year affected by them.*

THE value of R in Chapter XVII. is an incomplete value, because, in deducing it, the inclination of the plane was neglected or supposed equal nothing. If we restore that neglected condition, we shall have

$$R = \frac{m'.(xx' + yy' + zz')}{r'^3} - \frac{m'}{\sqrt{[(x'-x)^2 + (y'-y)^2 + (z'-z)^2]}}.$$

Let x, x', be supposed to be measured along the intersection of the two orbits, y, y', in the plane of the orbit of m, and the angular distances, or longitudes v, v', in the planes of the respective orbits of the bodies m and m'; and, let the inclination of those planes be ϕ; then, for the purpose of converting R into a function of r, r', v, v', &c. we have

$$x = r\,.\cos.\,v \qquad x' = r'.\cos.\,v',$$

$$y = r\cdot\sin.\,v \qquad y' = r'.\sin.\,v'\cos.\,\phi,$$

$$z = 0 \qquad z' = r'\sin.\,v'\sin.\,\phi\,;$$

consequently,

$$xx' + yy' = rr'\cos.\,v\,.\cos.\,v' + rr'.\sin.\,v'.\sin.\,v\,.\cos.\,\phi$$

$$= rr'(\cos.\,v\,.\cos.\,v' + \sin.\,v\,.\sin.\,v') - rr'.\sin.\,v\sin.\,v'(1 - \cos.\,\phi),$$

(*Trig.* pp. 26, 36.)

$$= rr'.\cos.(v'-v) - 2rr'.\sin.\,v'.\sin.\,v\,.\sin.^2\frac{\phi}{2},$$

Again, the square of the denominator of the second term in the value of R equals

$$x^2 + y^2 + x'^2 + y'^2 + z'^2 - 2(xx' + yy'),$$

or

$$r^2 + r'^2 - 2rr'\cos.(v'-v) + 4rr'.\sin.\,v'\sin.\,v\,.\sin.^2\frac{\phi}{2}.$$

Hence, if we develop the second term in the value of R; (but, by reason of the smallness of $\sin^2.\frac{\phi}{2}$, not beyond the second term), we shall have

$$R = \frac{m'r}{r'^2}\cos.(v'-v) - \frac{2m'r}{r'^2}\sin.\,v'.\sin.\,v\cdot\sin^2.\frac{\phi}{2}$$

$$-\frac{m'}{\sqrt{[r'^2 - 2rr'\cos.(v'-v) + r^2]}} + \frac{2m'rr'\sin^2.\frac{\phi}{2}\sin.\,v'.\sin.\,v}{(r'^2 - 2rr'\cos.(v'-v) + r^2)^{\frac{3}{2}}}.$$

In which expression the first and third terms are those which are given and expanded in Chapter XVII. The terms arising from the developments of the second and fourth are those which, arising from the inclination of the plane, are necessary to complete the value of R.

The formula for $\frac{1}{(a'^2 - 2aa' \cos. \omega + a^2)^{\frac{3}{2}}}$, which in pp. 295, &c. was deduced for the conveniently determining of the differential coefficients $\frac{dA}{da}$, $\frac{dB}{da}$, &c. will now serve another purpose. Indeed, as it is plain, it is requisite for expanding the fourth term.

By means of it, then, and of the expanded forms of Chap. XVII. we may express R in a series of cosines of the mean motions, &c. and thence, immediately, the values of $\frac{dR}{de}$, $\frac{dR}{d\pi}$, &c.

But in this research of the value of R it is a very important point to determine whether it contains any constant quantities. If there should be such, involving either the inclination, or the nodes, or the perihelion, &c. then, (see p. 420.) some of the elements would necessarily have *secular* variations.

In order to determine these constant parts in the value of R we must extend its development beyond the forms of pp. 276, 277, 279, and include terms involving e^2, e'^2, ee': but, it is not proposed to include terms involving higher powers or products: for, it is a supposition, in this as indeed it has been in all preceding enquiries, that e^2, e'^2, ee', $\sin^2. \frac{\phi}{2}$ are very minute quantities. We will examine the terms of the expression of R in their order; and, it will be convenient to premise, that we shall be principally guided in this examination by looking after terms the factors of which contain the cosines of similar arcs: for, it is plain, (see *Trig.* form [7], p. 27.) that one of the terms of

$$\cos. (mt + a) \cos. (mt + b),$$

when expanded must be constant and equal to $\frac{1}{2} \cos. (a-b)$.

First Term, $\frac{m'r}{r'^2} \cos. (v' - v)$.

Make $p = 1$, and the sixth term of the value of $\cos. (v' - v)$, (see p. 275.) is constant and equal

$$e\,e'\cos.(\pi' - \pi),$$

but the first term of $\frac{r}{r'^2}$, when expanded, must be $\frac{a}{a'^2}$; there is, therefore, on this account, a constant term introduced equal to

$$\frac{m'a}{a'^2}\,e\,e'.\cos.(\pi' - \pi).$$

Again, see p. 278.

$$\frac{m'r}{r'^2} = \frac{m'a}{a'^2}\cdot\left\{\begin{array}{l} 1 - e\cos.(nt + \epsilon - \pi) + 2\,e'.\cos.(n't + \epsilon' - \pi') \\ - 2\,e\,e'.\cos.(nt + \epsilon - \pi).\cos.(n't + \epsilon' - \pi') \\ + \,\&\text{c}. \end{array}\right\}$$

Now the third and fifth terms of the value of $\cos.(v' - v)$ (see p. 275.), are

$$-e\cos.(n't + \epsilon' - \pi), \quad -e'.\cos.(nt + \epsilon - \pi'),$$

which combined, respectively, with the third and second terms of $\frac{m'r}{r'^2}$, produce these constant cosines,

$$-\frac{m'a}{a'^2}\,e\,e'.\cos.(\pi' - \pi), \quad \frac{m'a}{2\,a'^2}\,e\,e'.\cos.(\pi' - \pi),$$

thirdly, one of the cosines, resulting from developing the fourth term in the value of $\frac{m'r}{r'^2}$, is

$$-\frac{m'a}{a'^2}\,e\,e'.\cos.(n't - nt + \epsilon' - \epsilon - \pi' + \pi),$$

which combined with the first term of the value of $\cos.(v' - v)$, (see p. 275.) produces

$$-\frac{m'a}{2\,a'^2}\,e\,e'.\cos.(\pi' - \pi)$$

Now, (see ll. 4, 14, 20.) these four constant cosines destroy each other: consequently, at least up to terms that involve e^3, e'^3, $e^2 e'$, &c. there are no constant terms in $\frac{m'r}{r'^2}\cos.(v' - v)$.

$$\textit{Second Term}, \quad -\frac{2\,\mathrm{m'r}}{\mathrm{r'^2}}.\,sin.\,\mathrm{v'}.\,sin.\,\mathrm{v}\,.\,sin^2.\,\frac{\phi}{2},$$

$$\sin.\,v'.\,\sin.\,v = \frac{1}{2}\,[\cos.\,(v'-v) - \cos.\,(v'+v)],$$

we may, therefore, combine the value of cos. $(v'-v)$ (p. 275.) with the value of $-\frac{m'r}{r'^2}$ (see p. 410.) and deduce some terms involving the cosines of constant arcs $(\pi'-\pi)$: the coefficients, however, of such terms will, at the least, involve ee'. But quantities (see p. 409.) involving $\sin^2.\,\frac{\phi}{2}\,e\,e'$ are not to be taken account of.

$$\textit{Third Term}, \quad -\frac{\mathrm{m'}}{\sqrt{[\mathrm{r'^2} - 2\,\mathrm{r}\,\mathrm{r'}\cos.(\mathrm{v'}-\mathrm{v}) + \mathrm{r^2}]}}\,.$$

Now, (see p. 275.)

$$\frac{1}{\sqrt{[r'^2 - 2rr'\cos.(v'-v)+r^2]}} = \frac{1}{2}P + P'.\cos.(v'-v) + P''\cos.2(v'-v) + \&c.$$

and, see p. 276. ll. 5, 6, 10,

$$P = A + \frac{dA}{da}\Delta a + \frac{dA}{da'}\Delta a' + \frac{d^2A}{2.\,d\,a^2}(\Delta\,a)^2 + \frac{d^2A}{2.\,d\,a'^2}\,(\Delta\,a')^2 + \&c.$$

= (if we take account merely of the constant parts)

$$A + a.\frac{dA}{d\,a}.\frac{e^2}{2} + a'.\frac{dA}{da'}.\frac{e'^2}{2} + a^2.\frac{d^2A}{d\,a^2}.\frac{e^2}{4} + a'^2.\frac{d^2\,A}{d\,a'^2}.\frac{e'^2}{4} + \&c.$$

which multiplied into $-\frac{m'}{2}$ will form the constant part of the third term arising from $-\frac{m'}{2}P$. Again, with regard to $P'.\cos.\,(v'-v)$; the constant part of P' will combine with the constant part of cos. $(v'-v)$, which, see p. 275, is the sixth term of its value, and equal $e\,e'.\cos.\,(\pi'-\pi)$, and form a constant quantity. The constant part of P' (see p. 276.) is similar to the above value of P (l. 13.) and equal,

$$B + a.\,\frac{d\,B}{d\,a}.\frac{e^2}{2} + \&c.$$

but, for reasons already stated (see p. 409. l. 17.) we need only reserve the first term B, which multiplied into $-m'ee'\cos.(\pi'-\pi)$ produces

$$-m'Bee'\cos.(\pi'-\pi).$$

In deducing the other constant quantities of $P'\cos.(v'-v)$ we must proceed on the principle laid down in p. 409. l. 22, &c. Now the third term $[-e.\cos.(n't+\epsilon'-\pi)]$ of $\cos.(v'-v)$ combined with the third term $\left(\frac{dB}{da'}\Delta a'\right)$ of P' (see p. 276.) when in such term $\Delta a'$ is expressed by its first term, namely, $-a'e'\cos.U$, or $-a'e'\cos.(n't+\epsilon'-\pi')$ produces one constant term, which is

$$\frac{a'e'e}{2}\cdot\frac{dB}{da'}\cdot\cos.(\pi'-\pi):$$

and $-e'.\cos.(nt+\epsilon-\pi')$, the fifth term in the value of $\cos.(v'-v)$ (see p. 275.) combined, similarly to the above combination, with $\frac{dB}{da}\Delta a$, produces

$$\frac{aee'}{2}\cdot\frac{dB}{da}\cos.(\pi'-\pi).$$

Lastly, $\cos.(n't-nt+\epsilon'-\epsilon)$, the first term of $\cos.(v'-v)$, combined with $\frac{d^2B}{da.da'}.\Delta a.\Delta a'$ (the sixth term of the value of P'), when, instead of Δa, $\Delta a'$, their first terms

$$-ae\cos.(nt+\epsilon-\pi),\quad -a'e'.\cos.(n't+\epsilon'-\pi'),$$

are written, produces

$$\cos.(n't-nt+\epsilon'-\epsilon)\times\frac{aa'ee'}{2}\cos.(n't-nt+\epsilon'-\epsilon-\pi'+\pi)+\&c.$$

the constant part of which is

$$\frac{aa'ee'}{4}\cos.(\pi'-\pi).$$

These three last terms (ll. 12, 16, 24.) then being multiplied by $-m'$: the sum of the constant parts of $-m'P'\cos.(v'-v)$ is

$$-m'\left(B+\frac{a}{2}\cdot\frac{dB}{da}+\frac{a'}{2}\cdot\frac{dB}{da'}+\frac{aa'}{4}\cdot\frac{d^2B}{da.da'}\right)ee'\cos.(\pi'-\pi).$$

There are no constant terms, within the prescribed limits (see p. 409.) to be derived from $P' \cos. (2v' - 2v)$, $P' \cos. (3v' - 3v)$, &c. $\cos. (2\,v' - 2\,v)$ (see p. 275.) contains no constant quantity: and the first constant quantity produced on the principle of p. 409. l. 22, &c. is by the combination of its first term, namely, $\cos. (2\,n'\,t - 2\,n\,t + 2\,\epsilon' - 2\,\epsilon)$ with $\frac{d^2 B}{da \,.\, da'} \,.\, \Delta\,a \,.\, \Delta\,a'$, when for Δa, $\Delta\,a'$, the terms $- \frac{a\,e^2}{2} \cos. 2U$, $- \frac{a'\,e'^2}{2} \cos. 2U'$, (see p. 276.) are substituted: but then the coefficient of the resulting term would involve $e^2\,e'^2$.

Fourth Term, $\dfrac{2\,m'\,r\,r' \,.\, sin.\, v' \,sin.\, v \,.\, sin^2 . \frac{\phi}{2}}{(r'^2 - 2\,r\,r'\, cos.\, (v' - v) + r^2)^{\frac{3}{2}}}$.

$$\frac{1}{(r'^2 - 2\,r r' \cos. (v' - v) + r^2)^{\frac{3}{2}}} = Q + Q' . \cos. (v' - v) + \&c.$$

$$= \frac{1}{2} A' + \frac{d\,A'}{d\,a} \Delta\,a + \&c.$$

$$+ (B' + \&c.) \cos. (n'\,t - n\,t + \epsilon' - \epsilon) + \&c.$$

$$+ \&c.$$

Now, $\sin. v' . \sin. v = \frac{1}{2} [\cos. (v' - v) - \&c.]$

$$= \frac{1}{2} \cos. (n'\,t - n\,t + \epsilon' - \epsilon) - \&c.$$

and the first term combined with

$$B' . \cos. (n'\,t - n\,t + \epsilon' - \epsilon) \text{ equals } \frac{B'}{4}:$$

the only constant term then, of which it is necessary to take account, is

$$2\,m'\,a\,a' \frac{B'}{4} \sin^2 \frac{\phi}{2}.$$

If F, then, be used to designate the constant part of R, we have

$$F = -\frac{m' A}{2}$$

$$-\frac{m'}{4}\left(a.\frac{dA}{da}e^2 + a'\frac{dA}{da'}e'^2 + \frac{a^2}{2}.\frac{d^2A}{2.da^2}e^2 + \frac{a'^2}{2}.\frac{d^2A}{da'^2}e'^2\right)$$

$$-m'\left(B + \frac{a}{2}.\frac{dB}{da} + \frac{a'}{2}.\frac{dB}{da'} + \frac{aa'}{4}.\frac{d^2B}{da.da'}\right)ee' \cos.(\pi' - \pi)$$

$$\ast + m'\frac{aa'}{2}B'.\sin^2.\frac{\phi}{2},$$

in which expression the quantities A, $\frac{dA}{da}$, B', &c. may be computed by the methods of Chapter XVIII.

It follows immediately, from the preceding expression and the formulæ of p. 404, that the axis-major is subject to no *secular* variation, but that the perihelion and node and eccentricity are: but, before we more fully consider this subject, we will, by means of the formulæ of Chapter XVIII, reduce F to a much more simple expression.

First, Reduction of $a.\frac{dA}{da} + \frac{a^2}{2}.\frac{d^2A}{da^2}$,

by p. 298. $\frac{dA}{da} = \frac{Aa}{a'^2 - a^2} - \frac{Ba'}{a'^2 - a^2}$;

$\therefore$ $\frac{d^2A}{da^2}$ (by substituting for the value of $\frac{dB}{da}$, p. 298, l. 12.)

$$= \frac{2Aa^2}{(a'^2 - a^2)^2} + \frac{Ba'^3 - 3Ba^2a'}{a.(a'^2 - a^2)^2};$$

consequently,

$$a.\frac{dA}{da} + \frac{a^2}{2}.\frac{d^2A}{da^2} = \frac{Aa^2a'^2}{(a'^2 - a^2)^2} - \frac{B(aa'^3 + a^3a')}{2.(a'^2 - a^2)^2},$$

$$\text{(see p. 297.)} = \frac{B'aa'}{2}.$$

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* ϕ (see p. 407.) is the mutual inclination of the planes of the orbits of m and m': but as we mean to employ that and similar symbols to designate the inclinations of the planes of the orbits m, m', &c. to a fixed plane, such as that of the ecliptic, we shall hereafter, in the expression for F, write I instead of ϕ.

Again, $a'.\frac{dA}{da'} + \frac{a'^2}{2}.\frac{d^2A}{da'^2} = a.\frac{dA}{da} + \frac{a^2}{2}.\frac{dA}{da^2}$,

for, by the nature of the trinomial $\frac{1}{\sqrt{(a'^2 - 2aa'\cos.\omega + a^2)}}$,

$$a.\frac{dA}{da} + a'.\frac{dA}{da'} = -A \ldots\ldots\ldots\ldots\ldots(1)$$

$$\therefore a'.\frac{d^2A}{da'^2} = \frac{2A}{a'} + \frac{2a}{a'}.\frac{dA}{da} - a.\frac{d^2A}{da.da'};$$

but from (1),

$$\frac{a'.d^2A}{da.da'} = -2.\frac{dA}{da} - a.\frac{d^2A}{da^2};$$

$$\therefore a'.\frac{dA}{da'} + \frac{a'^2}{2}.\frac{d^2A}{da'^2} = a.\frac{dA}{da} + \frac{a^2}{2}.\frac{d^2A}{da^2}.$$

Hence, the second line in the preceding value of F equals

$$-\frac{m'}{8}B'aa'(e^2 + e'^2).$$

Secondly, Reduction of $\mathrm{B} + \frac{\mathrm{a}}{2}.\frac{\mathrm{dB}}{\mathrm{da}} + \frac{\mathrm{a}'}{2}.\frac{\mathrm{dB}}{\mathrm{da}'} + \frac{\mathrm{aa}'}{4}.\frac{\mathrm{d^2B}}{\mathrm{da.da}'}$,

by p. 298, $\frac{dB}{da} = \frac{Aa'}{a'^2 - a^2} - \frac{Ba'^2}{a.(a'^2 - a^2)}$;

$$\therefore \frac{d^2B}{da.da'} = \frac{a'}{a'^2 - a^2}.\frac{dA}{da'} - \frac{a'^2}{a.(a'^2 - a^2)}.\frac{dB}{da'}$$
$$- \frac{A.a'^2 + a^2}{(a'^2 - a^2)^2} + \frac{2Baa'}{(a'^2 - a^2)^2},$$

which, by virtue of the equation of p. 298, l. 12, and of the equation of condition,

$$a\frac{dB}{da} + a'.\frac{dB}{da'} = -B,$$

is equal

$$-A.\frac{a^2 + a'^2}{(a'^2 - a^2)^2} + \frac{2Baa'}{(a'^2 - a^2)^2}.$$

Multiply this last equation by $\frac{a\,a'}{4}$, and add it to

$$B + \frac{a}{2}.\frac{d\,B}{d\,a} + \frac{a'}{2}.\frac{d\,B}{d\,a'} = \frac{B}{2},$$

and the sum equals

$$-\frac{A\,a\,a'.(a'^2+a^2)}{4.(a'^2-a^2)^2} + \frac{B}{2}.\frac{a'^4-a^2\,a'^2+a^4}{(a'^2-a^2)^2}.$$

But from the formula [α] of p. 295, bottom line,

$$C' = \frac{2\,B'.(a'^2+a^2)}{a\,a'} - 3\,A'$$

$$= A\,\frac{a'^2+a^2}{(a'^2-a^2)^2} - 2\,B\,\frac{a'^4-a^2\,a'^2+a^4}{a\,a'\,(a'^2-a^2)^2},$$

by substituting for B', A', their values such as are given in p. 297. Hence, the former sum (l. 4.) equals

$$-\frac{C'\,a\,a'}{4};$$

the third line, therefore, of the preceding value of F, (see p. 414.) equals

$$\frac{m'\,C'\,a\,a'}{4}.e\,e'.\cos.(\pi'-\pi).$$

Hence, we obtain this simple and convenient expression (see pp. 414, 415, and Note to p. 414.)

$$F = -\frac{m'}{2}\left\{\begin{array}{l} A + \frac{B'\,a\,a'}{4}\,(e^2+e'^2) \\ \quad - \frac{C'\,a\,a'}{2}\,e\,e'\cos.(\pi'-\pi) \\ \quad - B'\,a\,a'.\sin.^2\frac{I}{2}. \end{array}\right\}$$

Let us now revert to the formulæ of p. 404, and examine what they become when they express the *secular* variations.

First formula;

$$\delta a = -2a^2 . dR, \text{ or, } = -2a^2 . \frac{dR}{d\epsilon} n\, dt = -2a^2 . \frac{dF}{d\epsilon} n\, dt.$$

Now the preceding value of F does not involve the arbitrary quantity ϵ: consequently, $\frac{dF}{d\epsilon} = 0$, or, the axis-major is subject to no *secular* variation.

Second; since $\frac{dF}{d\epsilon} = 0$,

$$\delta e = \frac{a}{e} \surd(1 - e^2) . \frac{dF}{d\pi} . n\, dt$$

$$= \frac{a}{e} . \frac{dF}{d\pi} n\, dt, \text{ nearly,}$$

$$= \frac{m' a^2 a' C' e'}{4} \sin. (\pi' - \pi) . n\, dt$$

$$= \frac{m'}{4} . \frac{a^2}{a'^2} . C' a'^3 . e' \sin. (\pi' - \pi)\, n\, dt.$$

Third; $\delta \pi = -\frac{a}{e} . \frac{dF}{de} . n\, dt.$

$$= \frac{m'}{4} a^2 a' \left(B' - C' \frac{e'}{e} \cos. (\pi' - \pi) \right) n\, dt.$$

Fourth; $\delta \epsilon = 2a^2 . \frac{dF}{da} n\, dt - \frac{a}{e} . \frac{dF}{de} n\, dt$, nearly,

the latter term has been already (see l. 10.) determined, and of the former the factor,

$$\frac{dF}{da} = -\frac{m'}{2} \left\{ \begin{array}{l} \frac{dA}{da} + \frac{dB'}{da} . \frac{aa'}{4} (e^2 + e'^2) \\ -\frac{dC'}{da} . \frac{aa'}{2} e e' \cos. (\pi' - \pi) \\ -\frac{dB'}{da} . a a' . \sin^2 . \frac{I}{2} \\ + \frac{B'a'}{4} (e^2 + e'^2) - \frac{C' a'}{2} ee' . \cos. (\pi' - \pi) - B'a' \sin.^2 \frac{I}{2} \end{array} \right\}$$

which may be farther reduced by substituting for the differential coefficients $\frac{dA}{da}$, $\frac{dB'}{da}$, &c. their values as expressed by, or immediately deducible from, the formulæ of Chap. XVIII.

Fifth; $\delta\gamma = \frac{1}{h\gamma}\cdot\frac{dF}{d\theta}dt$

$$= \frac{m'}{2h\gamma}B'aa'\frac{d.\left(\sin^2.\frac{I}{2}\right)}{d\theta}$$

$$= -\frac{m'B'aa'}{4h\gamma}\cdot\frac{d.(\cos.I)}{d\theta}, \quad \text{since } 2\sin^2.\frac{I}{2} = 1 - \cos.I.$$

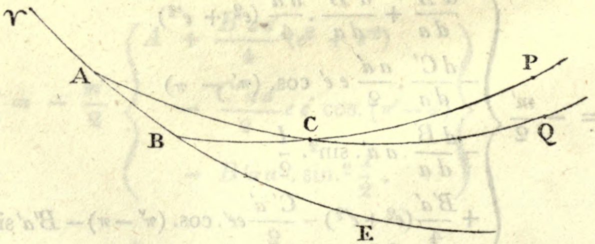
We must, therefore, find an expression for cos. I in terms of the sine and cosine of θ.

Let, in the subjoined figure,

$\phi = \angle PBE$, the inclination of PCB to the ecliptic,
$\phi' = \angle QAE$,
$I = \angle PCQ$, or $\angle ACB$ the inclination of the two orbits,
$\theta = \Upsilon B$, the longitude of the node of PCB.
$\theta' = \Upsilon A$, the longitude of the node of QCA, and consequently,
$\theta - \theta' = AB$;
then we have (see *Trig*. Chap. XI. p. 146.) in the spherical triangle ABC,

$$\cos.(\theta-\theta')\sin.(180^\circ-\phi)\sin.\phi' = \cos.I + \cos.(180^\circ-\phi)\cos.\phi',$$

or,



$$\cos.I = \sin.\phi\,\sin.\phi'.\cos.(\theta-\theta') + \cos.\phi\,\cos.\phi'.$$

Hence,

$$\frac{d\,(\text{cos. } I)}{d\,\theta} = -\,\text{sin. } \phi\,.\,\text{sin. } \phi'.\,\text{sin. } (\theta - \theta'),$$

and

$$\delta\,\gamma = \frac{m'\,B'\,a\,a'}{4\,h\,.\,\gamma}\,.\,\text{sin. } \phi\,.\,\text{sin. } \phi'.\,\text{sin. } (\theta - \theta').$$

But since, by supposition, the inclinations ϕ, ϕ' are very small, $d\,\gamma = d\,\phi$, nearly, and $\gamma = \text{sin. } \phi$: accordingly,

$$\frac{\delta\,\phi}{d\,t} = \frac{m'\,B'\,a\,a'}{4\,\sqrt{a}}\,.\,\text{sin. } \phi'\,.\,\text{sin. } (\theta - \theta')$$

$$= \frac{m'\,B'\,a^2\,a'}{4}\,n\,.\,\text{sin. } \phi'\,.\,\text{sin. } (\theta - \theta'),$$

or, very nearly,

$$= \frac{m'\,n}{4}\,B'\,a'^3\,.\,\frac{a^2}{a'^2}\,\text{tan. } \phi'\,.\,\text{sin. } (\theta - \theta').$$

Sixth; $\dfrac{\delta\,\theta}{d\,t} = -\,\dfrac{1}{h\,\gamma}\,.\,\dfrac{d\,F}{d\,\gamma}$

$$= -\,\frac{1}{\sqrt{a}\,.\,\text{sin. } \phi}\,.\,\frac{d\,F}{d\,\phi},\ \text{nearly},$$

$$= -\,\frac{m'\,B'\,a\,a'}{4\,\sqrt{a}}\left(\text{cos. } \phi' - \frac{\text{cos. } \phi}{\text{sin. } \phi}\,\text{sin. } \phi'\,.\,\text{cos. } (\theta - \theta')\right)$$

$$=,\ \text{nearly},\ -\,\frac{m'\,B'\,a^2\,a'\,n\ \text{cos. } \phi'}{4}\left(1 - \frac{\text{tan. } \phi'}{\text{tan. } \phi}\,\text{cos. } (\theta - \theta')\right).$$

These are the expressions for the *secular* variations of the elements, and from which several interesting results may be obtained.

We will first turn our attention to the variation of the eccentricity, namely,

$$\delta\,e = \frac{m'\,a^2\,C'\,a'}{4}\,e'.\,\text{sin. } (\pi' - \pi)\,.\,n\,d\,t.$$

This is the variation produced in the eccentricity of the orbit of the planet whose mass is *m*, by the disturbing force of another planet whose mass is m'. But a planet is disturbed, more or less, by all the other planets: let m'' be the mass of a third planet, e'', the eccentricity of its orbit, π'' the longitude of its perihelion, a'' its mean distance, and supposing C' the coefficient of the third term of the development of

$$[a'^2 - 2a'a \cos.(v' - v) + a^2]^{-\frac{3}{2}},$$

to be represented by $[a, a']$, the coefficient corresponding to C' in the development of

$$[a''^2 - 2a''a \cos.(v'' - v) + a^2]^{-\frac{3}{2}},$$

or the coefficient of cos. $2.(v'' - v)$, may conveniently be represented by $[a, a'']$; in which case, the variation arising from the disturbing force of the body whose mass is m'', will be

$$\frac{m'' a^2}{4} [a, a''] a'' e''. \sin.(\pi'' - \pi) n\, d t,$$

and similar expressions will represent the perturbations of bodies of which m''', m^{iv}, &c. should be the masses: so that the whole variation of the eccentricity (which is the aggregate of the partial variations) will be thus expressed,

$$\delta e =$$

$$\frac{n a^2}{4} d t \left[m'a'e'.[a, a'] \sin.(\pi' - \pi) + m'' a'' e'' [a, a''] \sin.(\pi'' - \pi) + \&c.\right]$$

In order to find the variation ($\delta e'$) produced in the eccentricity (e') of the planet m' * by the planet m, we must, in the preceding expression, write a', for a, e for e', m' for m, &c. but C', since it will still be the coefficient of the third term of the development of the above trinomial, (see l. 6.) will remain unaltered: and, accordingly,

$$\delta e' = \frac{m a'^2 C' a}{4} e . \sin.(\pi - \pi') n' d t,$$

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* For the sake of abridgment we call the planets m, m', m'', &c. those of which the masses are m, m', m'', &c.

or, if we choose to use a symbol* analogous to that of p. 420, l. 7.

$$\delta e' = \frac{m\,n'\,a'^2\,a\,e}{4}\,d\,t\,.\,[a', a]\ \text{sin.}(\pi - \pi').$$

Let us confine our attention, for a moment, to the case of two planets, Jupiter and Saturn, for instance: since

$$\delta e = \frac{m'\,a^2\,C'\,a'\,e'}{4}\ \text{sin.}\ (\pi' - \pi)\,n\,d\,t,$$

$$\text{and}\ \delta e' = -\frac{m\,a'^2\,C'\,a\,e}{4}\ \text{sin.}\ (\pi' - \pi)\,n'\,d\,t.$$

If π', the longitude of Saturn's perihelion, be greater than π the longitude of Jupiter's (which it is, since see *Astron.* p. 284. $\pi' - \pi = 78^0$, nearly,) δe is positive and $\delta e'$ negative: and, as it is plain from the two expressions of p. 421, ll. 5, 6. as long as the eccentricity of Jupiter's orbit is increased by the action of Saturn, so long will the eccentricity of Saturn's be diminished by the action of Jupiter. If π', and π remained strictly invariable, or, received, each, either an equal increment or equal decrement, the contemporaneous augmentations and diminutions of the eccentricities $(\delta e, -\delta \epsilon')$ caused by the mutual perturbation of Jupiter and Saturn, would for ever continue; the variations would be truly secular; Saturn's orbit would at length become circular, and Jupiter's an elongated ellipse or oval. But the perihelia are neither stationary nor equally progressive: so that it is no consequence of the preceding expressions that the eccentricity of Jupiter's orbit, if at any epoch it were increased by Saturn's disturbing force, would for ever continue to be so increased.

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* There is no convenience whatever in these symbols if we want merely to compute the mutual perturbations of two planets. But they are very convenient when it is necessary to express, by means of formulæ, the several inequalities produced in the elements of one planet, by the respective actions of all the other planets. We may see this by the instance in the text. If the symbol C' be used when a planet m' disturbs another m, there are no convenient symbols to be found which shall correspond to C' in the cases of planets m'', m''', &c. disturbing m: but, $[a, a']$ being once explained, $[a, a'']$, $[a, a''']$, &c. explain, as it were, themselves. Since they stand for formulæ or series similar to that which $[a, a']$ is written for; or are expounded by those very formulæ, &c. when, instead of a', a'' and a''', &c. are respectively written. M. Lagrange in his Memoirs on '*Secular Variations*' in the *Berlin Acts* for 1781, 1782, and M. Laplace on the same subject in the *Acad. des Sciences* for 1785, pp. 76, &c. and in the *Mec. Cel.* Chap. VII. Liv. 2, have used a notation founded on similar principles: thus $\boxed{0,\ 1}$ represents $\frac{m'\,n\,a^2\,a'}{4}\,C'$, the planet m' being the disturbing body: analogously, therefore, $\boxed{0,\ 2}$ $\boxed{0,\ 3}$ &c. will represent similar quantities to $\frac{m'\,n\,a^2\,a'}{4}\,C'$, when the same planet is disturbed by other planets m'', m''', &c. And $\boxed{1,\ 0}$ $\boxed{2,\ 0}$, &c. will represent quantities similar to $\frac{m'\,n\,a^2\,a'}{4}\,B'$, when the planets m', m'', &c. are disturbed, respectively by the planet whose mass is m.

But it is easy to shew, after the following manner, that the variations of the eccentricities are confined within certain limits.

Since, $an = \frac{a}{a^{\frac{3}{2}}} = \frac{1}{\sqrt{a}}$, and $a'n' = \frac{1}{\sqrt{a'}}$: if we multiply δe $\delta e'$ (see p. 421.) by $em\sqrt{a}$, $e'm'\sqrt{a'}$, respectively, and add the results, we have

$$m\sqrt{a}\,.\,e\,\delta e + m'\sqrt{a'}\,.\,e'\,\delta e' = 0,$$

whence,

$$m\sqrt{a}\,.\,\frac{e^2}{2} + m'\sqrt{a'}\,.\,\frac{e'^2}{2} = K,$$

where K is a constant quantity: in order to compute it, we have for the epoch of 1801. See *Astron.* p. 284.

$$e = .048178,$$

$$e' = .056168,$$

and, see p. 329. of the present Treatise,

$$m = \frac{1}{1067.09},$$

$$m' = \frac{1}{3359.4},$$

$$a = 5.20279,$$

$$a' = 9.53877,$$

whence

$$K = .00001243.$$

Now this being the value of K, e can never exceed a certain limit, for, since

$$m\sqrt{a}\,.\,e^2 + m'\sqrt{a'}\,.\,e'^2 = .00002486,$$

e can become larger only by the diminution of e', and can, at the most, never exceed that value which will result from the preceding equation when $e' = 0$: in which case

$$e = .06065.$$

so that we are at least certain the eccentricity can never go beyond this quantity.

In like manner we may prove that e' can never be so far increased as to reach the limit .09247. But the real limits will be within those which have been assigned. When the eccentricity of Jupiter is greatest, that of Saturn, as it is plain from the formula of l. 3, will be the least; and reversely. The corresponding augmentations and diminutions will have the same period, which, according to Lagrange, will exceed 35200 years *.

The preceding demonstration, as we shall presently see, may be extended farther. The change of the eccentricity of the orbit of any one planet arising from the perturbations of the other planets is always limited.

In order to compute δe, $\delta e'$, the value of $C'a'^3$, must be previously computed by the method of Chapter XVIII. since $\frac{a}{a'} = .545317$. And, by that method, $C'a'^3 = 2.0821$: whence,

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* This result, with other results like those in the text, but obtained by different methods, is to be found in the *Berlin Memoirs* for the years 1781, 1782.

$$\frac{\delta e}{dt} = 0''.276, \text{ and } \frac{\delta e'}{dt} = -\,0''.55\,*.$$

These are the variations of the eccentricity, and since, in orbits of small eccentricity, (and such are the orbits of the planets) the eccentricity is half the greatest *equation of the centre* † (see *Astron.* p. 203.) we have the variations of the greatest equations of the centre of Jupiter and Saturn represented by 0″.55, and − 1″.1 respectively; and their variations for 100 years, (which are sometimes called their *secular* variations) are 55″, 1′ 50″, respectively.

The changes which the eccentricities of the orbits of Jupiter and Saturn suffer from the other planets are very inferior to those which are produced by their mutual perturbations.

But by far the most interesting result deducible from the preceding formula is that which relates to the variation of the eccentricity of the Earth's orbit. For it serves to explain (see pp. 226. and *Astron.* p. 312.) a phenomenon which long em-

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* *Computation.*

For δe (see p. 421.)		for $\delta e'$ (see p. 421.)	
Logarithms.		Logarithms.	
$\frac{a^2}{a'^2}$	9.4734	$\frac{a}{a'}$	9.7366
m'	6.4737	m	6.9717
e'	8.7495	e	8.6828
n	5.0384	n'	4.6434
sin. 78°	9.9904	sin. 78°	9.9904
$\frac{C'a^3}{4}$	9.7164	$\frac{C'a'^3}{4}$	9.7164
	9.4418 = log. 276.		9.7413 log. 551.

† The eccentricity itself is not an object of observation, but is a result deduced from the greatest equation of the centre (see *Astronomy*, p. 265.).

barrassed Astronomers, namely, the *acceleration* of the Moon's mean motion.

The *acceleration* is not, strictly speaking, a phenomenon : it is rather an anomalous result from the comparison of observations ; and it may be thus deduced. Compare together, for the purpose of determining the mean motion, the observations of 1680 and 1750, and then those of 1750 and 1800. The results will be found to differ : they will differ still more, if modern observations are compared with the observations of Ptolemy and Hipparchus. As you approach modern times the Moon's mean motion seems accelerated : and, therefore, capable of being represented by

$$n\,t + A\,t^2,$$

in which $n\,t$ should be the mean motion and $A\,t^2$ its *acceleration.*

Now this anomalous fact of the acceleration of a mean motion may be accounted for, on Newton's principles, from the *variation* of the eccentricity of the solar orbit.

If we refer to p. 226. we shall find that (n denoting the Moon's mean motion),

$$n\,t = v + \frac{3\,m^2}{2}\int e'^2.d\,v + \&c.$$

Now the second term, if e', the eccentricity of the solar orbit, were strictly invariable, would equal $\frac{3\,m^2}{2}e'^2\,v$, in which case we should have

$$n\,t = v\left(1 + \frac{3\,m^2}{2}e'^2\right) + \&c.$$

and, the other terms of the value of $n\,t$ being *periodical,* there would be in the value of nt no term expounding a secular equation.

But if e' should vary, then (making $d\,v = n\,d\,t$ and correcting the integral),

$$\frac{3\,m^2}{2}\int e'^2.n\,d\,t = \frac{3\,m^2}{2}.\int(e'^2 - E'^2)\,n\,d\,t,$$

E' being the value of e' at a particular epoch when t, then representing the commencement of time, is $= 0$; and, if e' should equal $E' - at + bt^2$, &c. $e'^2 - E'^2$ would equal

$$2E'at + (a^2 - 2bE')t^2 + \&c.$$

and consequently, $\frac{3m^2}{2}\int(e'^2 - E'^2)\,n\,dt$ would be represented by a series such as

$$At^2 + Bt^3 + \&c.$$

and, (from the minuteness of B and the coefficients of t^3, t^4, &c.) nearly, by the first term At^2, which, in that case, would expound a secular equation: such as would serve to correct (see pp. 324, 342.) the *uniform acceleration* of the mean motion.

This is some advance towards explanation; in order to arrive at it, we must shew, on the principles of this and the preceding Chapter, that the eccentricity of the Earth's orbit varies, that it *decreases*,* and (which is the main point) that the quantity of its

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* It is essential to the explanation that the eccentricity should decrease: for an augmentation of eccentricity, on Newton's principles, would account for a *retardation* of the Moon's mean motion: let t, t', t'', denote three periods, of 40, 80 and 120 years, for instance, reckoned from the epoch of 1700, and let v, v', v'' be the corresponding longitudes: then, if abstraction be made of all periodical inequalities, we have

$$nt'' = v'' - At''^2,$$
$$nt' = v' - At'^2,$$
$$nt = v - At^2,$$
$$\therefore n = \frac{v'' - v'}{t'' - t'} - A.(t'' + t')$$
$$= \frac{v' - v}{t' - t} - A(t' + t).$$

Now, $\frac{v'' - v'}{t'' - t'}$, $\frac{v' - v}{t' - t}$, would be used to represent the mean motion (n), on the supposition that there was no secular equation: the first fraction, therefore, would be larger than the second: or, (supposing decrease is such as will make the computed accord with the observed *acceleration.*

From the formula of p. 417, it appears that the eccentricity of the Earth's orbit, like that of any other orbit, must vary, except, (which is very unlikely) the mutual perturbations of the planets should exactly balance one another. Venus augments, the other planets diminish, the eccentricity; and the diminutions exceed the augmentation: but the fact and quantity of the excess will be at once shewn by the formula of p. 417. and by a computation exactly similar to that by which in p. 424. the eccentricities of the orbits of Jupiter and Saturn were computed. The result of the computation is

Diminution of Eccentricity.		Augmentation of Eccentricity.	
From Mercury....	$-\ 0''.0040$		
Mars	$-\ 0.0246$		
Jupiter......	$-\ 0.0798$		
Saturn	$-\ 0.0003$	from Venus	$+\ 0''.0152$
	$-\ 0''.1087$	....	$-\ 0.1087$
			$-\ .0935$

The annual diminution, therefore, of the Earth's eccentricity is $-0''.0935$, and, of the greatest equation of the centre, $-0''.187$. The *secular* (see p. 424.) are $-\ 9''.35$, $-\ 18''.7$, respectively.

Hence, since $\frac{\delta e'}{d t} = -\ 0''.0935$, we have, very nearly,

$$e' = E' - 0''.0935 \times t,$$

whence,

$$\frac{3}{2}\, m^2 \int (e'^2 - E'^2)\, n\, d t = .001018 . t^2,$$

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posing the above equations to be true) n would result a larger quantity from the observations of 1820 and 1780, than from those of 1780 and 1740: but the contrary would happen if

$$n t = v + A t^2,$$

which would be the equation if e' were greater than E'.

t being the number of years elapsed from the given epoch 1700. Let i denote the number of centuries from the same epoch, then, since $t = 100 \times i$, we have the above secular equation equal to

$$10''.18 \times i^2.$$

If the above computation be more accurately made, and account be taken of the terms involving t^3 and i^3, then, according to Laplace and Delambre, the above secular equation is

$$10''.18162\,i^2 + 0''.018538\,i^3.$$

Now the above coefficient of the secular equation computed, as we have seen, on Newton's Principle and Law of Gravitation, agrees very nearly with observation : and thence we derive a very strong presumption that the cause of the Moon's acceleration is rightly assigned to the diminution of the eccentricity of the Earth's orbit; the diminution itself being caused by the disturbing forces of the planets.

There is in this explanation of the phenomenon of the Moon's acceleration a strong proof of the truth of the Law of Gravity; and the proof is of a refined kind: for, the perturbations of the planets are not communicated immediately to the Moon, but transmitted by means of the Earth. The *acceleration* is, as it has been called, a *reflected* effect. The immediate and direct effect of the disturbing forces of the planets is a diminution of the eccentricity of the Earth's orbit: the secondary and collateral effect is the augmentation of the Moon's mean velocity. This proof of the truth of Newton's System well merits attention: it is not, indeed, singular in *Astronomy*, but it is the first of the kind we have met with.

The reflected effect, we may also farther remark, is greater than the direct: in 2000 years the diminution of the eccentricity would not exceed (see p. 427.) 3′. 7″: whereas, in the same period, the Moon's mean motion would be increased nearly by 1° 11′: so that it is this latter, which is the indirect effect, that makes manifest the influence of the attraction of the planets on the dimensions of the Earth's orbit.

The proof, however, of the truth of the principle and Law of

Gravity, derived from the preceding explanation of the Moon's acceleration, is not altogether placed beyond the reach of doubt. For, the diminution of the eccentricity of the Earth's orbit, which after a long period is never large, is merely the accumulated excess of some very minute inequalities above others. The inequalities arise from the disturbing forces of the planets. Those forces depend, in part, on the masses of the disturbing planets, all which masses are not precisely known. The masses, for instance, of Venus and Mars, planets which have not satellites, are from that circumstance, uncertain. There is no formula (see p. 30.) for computing them: and it may be mentioned, as a kind of practical proof of the uncertainty which remains on this head, that the value of the mass of Venus has been under change and correction from the publication of the *Principia* to the present time. It has been differently assigned by Newton, Clairaut, Lalande *, and by the author of the *Mecanique Celeste.*

It happens, however, (fortunately we may say) that the diminution of eccentricity is principally caused by Jupiter, a planet whose mass is best known. The annual effect of Jupiter's disturbing force on the eccentricity is (see p. 427.) more than three times that of Mars, and more than five times that of Venus †. Now the quantity of matter in Jupiter may very accurately be determined from the period of his fourth satellite: and, if we adopt an argument like that which has just been used (see ll. 11, &c.) we may infer that Jupiter's mass is exactly determined, because it has been expressed by nearly the same fraction both by Newton and Laplace. The former represents it by $\frac{1}{1067}$ (see Prop. VIII. Lib. 3.) the latter by $\frac{1}{1067.09}$.

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* There cannot be a stronger proof of the uncertainty in which Astronomers were formerly with regard to Venus's mass than the following passage from Lalande's Memoir in the *Acad. des Sciences,* 1750, p. 361. 'Or il me paroît probable que la masse de Venus est en effet *double,* ou a peu pres, de celle que je supposai avec M. Euler dans mon Premiere Memoire.

† This reasoning is not quite exact, but it is not altogether wrong. From inferences drawn from the comparison of many and various observations the masses of Venus and Mars are now known within certain limits.

The cause, then, of the Moon's acceleration, is, *probably*, rightly assigned. The coefficient of the secular equation, with which it is necessary to correct the Moon's mean motion, is nearly the same whether it be computed by theory or from observation. But an equation such as

$$10''.18\, i^2 + 0''.0185\, i^3,$$

would perpetually increase with the time. It might be useful, indeed, in the formation of Tables, but it would seem to establish the *anomalous* fact of the Moon's acceleration: anomalous inasmuch as all other mean motions in the system are invariable: and, if so, a fact very curious and interesting, since then the Moon, however slow its approach, would be perpetually drawing nearer and nearer to the Earth. But as in the case of the eccentricity of Saturn's orbit (see p. 422.) so in this of the Earth's; the diminution of the eccentricity from the disturbing forces of the planets will not exceed a certain limit: after it has reached that, an augmentation will take place from the very same disturbing forces acting under a change of circumstances: so that, the variation of the eccentricity is strictly a periodical inequality: the Moon's acceleration is of the same kind, and which will become a *retardation* when the disturbing forces of the planets shall have begun to augment the eccentricity of the Earth's orbit.

This diminution of the eccentricity of the Earth's orbit serves also to explain the retardations of the mean motions of the Lunar Perigee and Node (see p. 182, and *Astron.* Chap. XXXII.).

We will next consider the variation of the longitude of the perihelion; which is

$$\delta\pi = \frac{m'}{4} a^2 a' . \left(B' - C' \frac{e'}{e} \cos.(\pi' - \pi) \right) n\, d\, t,$$

$$\text{or } = \frac{m'}{4} . \frac{a^2}{a'^2} . \left(B' a'^3 - C' a'^3 . \frac{e'}{e} \cos.(\pi' - \pi) \right) n\, dt.$$

The immediate inference from this expression is, that the perihelia of the orbits of planets are not, by the effect of disturbing forces, necessarily *progressive.* They are progressive if B' be greater than $\frac{C'e'}{e} \cdot \cos. (\pi' - \pi)$.

If we refer, however, to the Tables of the elements of the orbits of planets (see *Astron.* pp. 284, &c.) it will be found that the perihelia of almost all the planets *progress.* Venus is an exception: the perihelion of her orbit *regresses* both from the action of the Earth which is without her orbit, and of Mercury[*], which is within. Their perturbations, in this case, exceed those of the other planets: each of which would, indeed, separately cause a *progression* of the perihelion of Venus's orbit.

The formula of p. 430. l. 30. expresses the variation of the perihelion of the orbit of a planet m from the action of another planet m'. Similar formulæ, as in the case of the variations of the eccentricity, (see p. 420.) will express the perturbations of the perihelion by planets m'', m''', &c.; and the sum of such formulæ will, as it is plain, expound the whole variation of the longitude of the perihelion, or the excess of its *progressions* above its *regressions.*

The formula for the variation of π' from the disturbing force of the planet m, may be obtained from the preceding by writing m instead of m', a instead of a', a' instead of a, &c. B' and C' will (see p. 420.) remain unaltered; so that

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[*] For the purpose of computing $\delta\pi$ in these two cases, we have (see *Astron.* pp. 284, 285.).

For Mercury's action.	For the Earth's action.
$\pi' = 74^\circ\ 21'$	$\pi' = 99^\circ . 30'$
$\pi = 128\ \ 37$	$\pi = 128\ \ 37,$
$e' = .205514$	$e' = .016853,$
$e = .006853$	$e = .006853,$
and, see Chap. XVIII.	
$B'a'^3 = 3.0353,$	$B'a'^3 = 8.8718,$
$C'a'^3 = 1.9505$	$C'a'^3 = 7.3865.$

In both of these cases $B' - C'\frac{e'}{e} \cos. (\pi' - \pi)$ is a negative quantity.

$$\delta\pi' = \frac{m}{4} a'^2 a \left(B' - C'\frac{e}{e'} \cos.(\pi - \pi')\right) n' d t,$$

$$\text{or } = \frac{m}{4} . \frac{a}{a'} \left(B' a'^3 - C' a'^3 . \frac{e}{e'} \cos.(\pi - \pi')\right) n' d t.$$

To illustrate these formulæ we may, as in the case of the eccentricities, take the instance of Jupiter and Saturn. The *progressions** of the perihelia of the orbits of these planets are derived, almost entirely, from their mutual perturbations.

1*st*. *The progression of Jupiter's perihelion computed from*

$$\frac{\delta\pi}{d t} = \frac{m' n}{4} . \frac{a^2}{a'^2} \left(B' a'^3 - C' a'^3 \frac{e'}{e} \cos.(\pi' - \pi)\right).$$

$B'a'^3$, $C'a'^3$, computed by the methods of Chapter XVIII. are, respectively, 3.1855, 2.0821, (see pp. 304, 423.).

First term computed.		Second term computed.	
	Logarithms.		Logarithms.
$\frac{a^2}{a'^2}$	9.4734		
m'	6.4737		
n.	5.0385		21.9855
$\frac{B'a'^3}{4}$	9.9011		
		e'	8.7495
	.8866 (= log. 7.702)		
		arith. comp. e	1.3171
		cos. 78°	9.3178
		$\frac{C'a'^3}{4}$	9.7164
			.0863 (=log. 1.2199).

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* Robison is mistaken when he *asserts,* in p. 385. of his *Mechanical Philosophy,* that 'the apsides of all the planets are observed to advance, *except those of Saturn, which sensibly retreat, chiefly by the action of Jupiter,*' and again, when he asserts, 'the apsides of the planet, discovered by Dr. Herschell, *doubtless retreat considerably,* by the great planets Jupiter and Saturn.' If the Author instead of referring his reader had himself referred to the Works which he quotes in the previous page (p. 383.) he would have found both reason and authority contradicting his assertions.

Hence, $\frac{\delta\pi}{dt} = 7''.702 - 1''.2199 = 6''.482.$

2dly, *The progression of Saturn's perihelion computed from*

$$\frac{\delta\pi'}{dt} = \frac{m\,n'}{4} \cdot \frac{a}{a'}\left(B'a'^3 - C'a'^3 \cdot \frac{e}{e'}\,cos.\,(\pi - \pi')\right).$$

Logarithms.		Logarithms.	
$\frac{a}{a'}$	9.7366	$\frac{C'a'^3}{4}$	9.7164
m	6.9717	....	21.3517
n'	4.6434	e	8.6828
$\frac{B'a'^3}{4}$ ...	9.9011	arith. comp. e'	1.2504
		cos. 78°	9.3178
(log. 17.898)	1.2528		.3191 (=log. 2.085).

Hence, $\frac{\delta\pi'}{dt} = 17''.898 - 2''.085 = 15''.813.$

In the preceding computation the values of π, π', e', &c. are such as belong to the epoch of 1800. Consequently, 6''.48, 15''.81, are the respective annual progressions of the perihelia of Jupiter and Saturn at that epoch: and if t be the number of years from that epoch, the corresponding quantities of *progressions* will be $6''.48\,t$, $15''.81\,t$.

The second term, $\frac{m'n}{4} \cdot \frac{a^2}{a'^2}\,C'a'^3\frac{e'}{e}\cos.\,(\pi' - \pi)$ varies from the changes in e', e, π', π. By reason of it, then, the mean progression is not constant: it will be increased by the diminution of e', and the augmentations of e and of $\pi' - \pi$.

If we wish to express the progressions in terms of their two parts, the constant and the variable, we have from the preceding computation,

$$\frac{\delta\pi}{dt} = 7''.702 - 5''.033 \times \frac{e'}{e} . \cos.\,(\pi' - \pi),$$

$$\frac{\delta\pi'}{dt} = 17''.898 - 11''.69 \times \frac{e}{e'} . \cos.\,(\pi' - \pi).$$

The values of $\frac{\delta\pi}{dt}, \frac{\delta\pi'}{dt}$, therefore, will be respectively equal to $7''.702$, $17''.898$, when $\pi - \pi' = 90^{\circ}$: the present value of which (an increasing value) is about 78°.

The mean values, $7''.702$, $17''.898$, are almost exactly in the proportion of 3 to 7, that is, (see p. 329.) of $m'\sqrt{a'}$ to $m\sqrt{a}$.

All the planets cause the perihelia of the orbits of Jupiter and Saturn to *progress*, but, as we have already said, by very minute quantities. In the case of Saturn their sum does not exceed the third of a second: in that of Jupiter, not one seventh.

We will now consider the expression for the variation of the inclination, which is, as it has been stated, (see p. 419.)

$$\frac{\delta\phi}{dt} = m' . \frac{B' a^2 a'}{4} n . \sin. \phi' \sin. (\theta - \theta'),$$

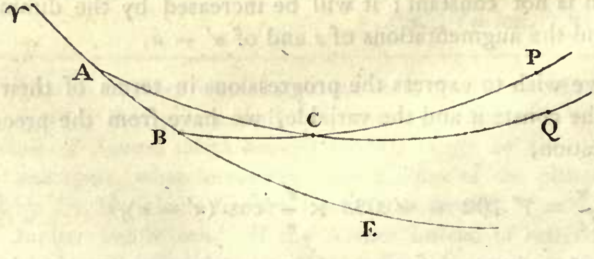
or $$= \frac{m' n}{4} B' a'^3 . \frac{a^2}{a'^2} \sin. \phi' . \sin. (\theta - \theta'),$$

or, ϕ' being very small, and $\tan. \phi' = \sin. \phi'$, nearly,

$$= \frac{m' n}{4} B' a'^3 . \frac{a^2}{a'^2} \tan. \phi' \sin. (\theta - \theta').$$

This is the variation produced by the disturbing force of a planet m'. But, as in the cases of the eccentricities and perihelia (see pp. 420, &c.) similar formulæ will obtain for the perturbations of the other bodies m'', m''', &c.

In order to find the effect, produced by the disturbing force of a body m, whose orbit is PCB, on the plane of the orbit QCA,



we must, as in the former cases, (see pp. 420, &c.) write m, a', &c. instead of m', a, &c. and, accordingly, we shall have

$$\frac{\delta \phi'}{d t} = \frac{m n'}{4} B' a'^3 . \frac{a}{a'} \sin. \phi . \sin. (\theta' - \theta),$$

ϕ, and ϕ' denoting the angles CBE, CAB.

If we multiply $\frac{\delta \phi}{d t}$ by $m n' \sin. \phi$, and $\frac{\delta \phi'}{d t}$ by $m' n \frac{a}{a'} \sin. \phi'$, we shall have

$$m n' \sin. \phi \frac{\delta \phi}{d t} + m' n . \frac{a}{a'} \sin. \phi' . \frac{\delta \phi}{d t} = 0,$$

or, since $n' = \frac{1}{a'^{\frac{3}{2}}}$, and $n = \frac{1}{a^{\frac{3}{2}}}$,

$$m \sqrt{a} . \sin. \phi . \delta \phi + m' \sqrt{a'} . \sin. \phi' . \delta \phi' = 0,$$

and consequently,

$$m \sqrt{a} . \cos. \phi + m' \sqrt{a'} . \cos. \phi' = K,$$

K being a constant quantity; or, making

$$k = \frac{m \sqrt{a} + m' \sqrt{a'} - K}{2},$$ we have

$$m \sqrt{a} . \sin^2 . \frac{\phi}{2} + m' \sqrt{a'} . \sin^2 . \frac{\phi'}{2} = k.$$

Inferences of a similar nature to those in pp. 422, &c. may be deduced from this formula: for, if ϕ, ϕ', be at any times small quantities, neither $\sin^2 . \frac{\phi}{2}$, nor $\sin^2 . \frac{\phi'}{2}$, can ever exceed certain limits: for instance, for the epoch 1800, (see *Astron.* 286.)

ϕ, the inclination of Jupiter's orbit, $= 1^\circ \; 18' \; 51''$,
ϕ', the inclination of Saturn's orbit, $= 2 \; 29 \; 34$,

therefore, (see p. 422.)

$$m \sqrt{a} . \sin^2 . 39' \, 25'' + m' \sqrt{a'} . \sin^2 . 1^\circ \, 14' \, 47'' = .000002263,$$

$$\text{or } k = .000002263.$$

Now if $\frac{\phi}{2}$ increase, it must, from the theorem of l. 12, increase by the diminution of $\frac{\phi'}{2}$: and it can, at the most, never exceed that value which it will have, from the above theorem, on making $\phi' = 0$: but in this extreme case the value of ϕ equals $2^\circ\ 5'\ 50''$, and, if it should reach that limit, it would subsequently regress from it; ϕ' from its lowest state would begin and continue to increase, and a sort of oscillation about a mean state of inclination would perpetually ensue.

The preceding result, which is one of the points of the *stability* of the planetary system, is, as we shall hereafter see, generally true. It has, indeed, been proved only of the variations of the inclinations of the planes of the orbits of Jupiter and Saturn arising from their mutual action; but these variations, like those of the eccentricities and perihelia, far exceed what the other planets are able to cause.

In order to compute the above-mentioned variations for the the epoch of 1800, we have (see *Astronomy*, p. 286.)

$$\begin{array}{lrrrr} & \theta' = & 111^\circ & 55' & 46'', \\ & \theta & 98 & 25 & 34, \\ \text{and therefore} & \theta' - \theta & 13 & 30 & 12. \end{array}$$

Hence, for ♃.	Logarithms.	For ♄.	Logarithms.
$\sin.\ (\theta' - \theta)$	9.3682		9.3682
$\sin.\ \phi'$ (see p. 435.)	8.6384	$\sin.\ \phi$	8.3604
$m' n \frac{a^2}{a'^2} . \frac{B' a'^3}{4}$ (p. 432.)	.8886	$m n' \frac{a}{a'}, B' \frac{a'^3}{4}$ (p. 433.)	1.2528
(log. .07856)	8.8952	(log. .09580)	8.9814

The respective annual variations, therefore, of the inclinations of the orbits of Jupiter and Saturn, from their mutual perturbations, are

$$-\ 0''.0785, \text{ and } 0''.0958,$$

and their secular variations

$$-\ 7''.85,\quad 9''.58.$$

There are several other consequences deducible from the preceding expressions of pp. 434, 435, but as they are, in some sort, connected with the motions of the nodes, we will now turn our attention to the expressions on which these latter motions depend. (See p. 419.)

$$\frac{\delta\,\theta}{d\,t} = -\,\frac{m'\,B'\,a^2\,d\,n}{4}\cos.\,\phi'\left(1 - \frac{\tan.\,\phi'}{\tan.\,\phi}\cos.\,(\theta - \theta')\right)$$

$$= -\,\frac{m'\,n}{4}\cdot\frac{a^2}{a'^2}\,B'\,a'^3.\left(1 - \frac{\tan.\,\phi'}{\tan.\,\phi}\cos.\,(\theta - \theta')\right),$$

if ϕ' be a small angle.

The above is an expression for the motion of the node on a fixed plane; such, for instance, as that of the ecliptic, at a given epoch. On such a plane, therefore, the nodes *regress* if $\frac{\tan.\,\phi'}{\tan.\,\phi}\cos.\,(\theta - \theta')$ be less than 1, and *progress* if the above fraction should be greater than 1.

$\delta\,\theta$ expresses the motion of the node of a body m by reason of the disturbing force of a body m'. In order to find $\delta\,\theta'$, the corresponding inequality from the action of the body m, make (see p. 420.) the usual alterations, and then

$$\frac{\delta\,\theta'}{d\,t} = -\,\frac{m n'}{4}\,.\,B'\,a'^3.\frac{a}{a'}\left(1 - \frac{\tan.\,\phi}{\tan.\,\phi'}\cos.\,(\theta' - \theta)\right).$$

Hence, since $\frac{\tan.\,\phi'}{\tan.\,\phi}\,.\cos.\,(\theta - \theta')$, and $\frac{\tan.\,\phi}{\tan.\,\phi'}\,.\cos.\,(\theta - \theta')$, may each be less than 1, the nodes both of Mars and Jupiter (taking that instance for illustration) may *regress*, on the plane of the ecliptic, from their mutual perturbations. But, if $\frac{\tan.\,\phi'}{\tan.\,\phi}\cos.\,(\theta - \theta')$ should be greater than 1, since that can only happen from tan. ϕ' being greater than tan. ϕ, $\frac{\tan.\,\phi}{\tan.\,\phi'}\cos.\,(\theta - \theta')$ must necessarily be less than 1. The inference, therefore, is of a different kind from the preceding (see l. 18.). If Jupiter's nodes *progress* (which is the case) from the disturbing force of Saturn, Saturn's nodes must necessarily *regress* from Jupiter's disturbing force. These are effects that take place, as we have observed, on the plane of the ecliptic, supposing it fixed: but (with regard to its mean and secular motions) the *node of the disturbed body invariably regresses on the orbit of the disturbing body*: for, if the plane of the orbit of the latter be that to which the motions of the former are referred (or which is the same thing, if we suppose the plane of the ecliptic coincident with that of the disturbing body), ϕ' and, consequently, tan. ϕ' must $= 0$, in which case,

$$\frac{\delta\theta}{dt} = -\frac{m'n}{4}\cdot\frac{a^2}{a'^2}\cdot B'a'^3,$$

and if the motions of the body m' be referred to the plane of the body m, then ϕ, and consequently, tan. $\phi = 0$, and

$$\frac{\delta\theta'}{dt} = -\frac{mn'}{4}\cdot\frac{a}{a'}\cdot B'a'^3.$$

Hence, the mean secular variations ($\delta\theta$, $\delta\theta'$) of the nodes are as

$$m'n\cdot\frac{a}{a'} \text{ to } mn',$$

or, as $m'\sqrt{a'}$ to $m\sqrt{a}$, (see pp. 328, 434.).

It is a consequence immediate from what precedes, that the nodes of the Moon's orbit must (*secularly*) regress on the plane of the ecliptic: since that is the plane of the orbit of the disturbing body, which is the Sun.

We will again use, for the exemplification of the preceding formulæ, (ll. 9, 12.) the instance of Jupiter and Saturn.

For Jupiter,		For Saturn,	
see p. 432.	Log.	see p. 433.	Log.
(a) $\frac{m'n}{4}\cdot B'a'^3\cdot\frac{a^2}{a'^2}$	.8886,	(b) $\frac{mn'}{4}B'a'^3\frac{a}{a'}$	1.2528;
$\therefore\ \frac{\delta\theta}{dt} = -7''.737$,		$\frac{\delta\theta'}{dt} = -17''.898$.	

These are the mean annual regressions of the nodes of Jupiter's and Saturn's orbits on the planes, respectively, of each other's orbit from their mutual perturbation. In order to find the regressions on the ecliptic from the same cause, we must compute

$$\frac{\tan.\ \phi'}{\tan.\ \phi} \cos.\ (\theta - \theta').$$

For the epoch of 1800, (see *Astron.* p. 286, and p. 438. of this Treatise),

$$\theta' - \theta = 13^{\circ}\ 30'\ 12'' \ldots\ldots \text{log. cos.} = 9.9878,$$

$$\phi = 1\ \ 18\ \ 51 \ldots\ldots\ldots \text{log. tan.} = 8.3606,$$

$$\phi' = 2\ \ 29\ \ 34 \ldots\ldots\ldots \text{log. tan.} = 8.6388\ ;$$

$$\therefore \log.\ \frac{\tan.\ \phi'}{\tan.\ \phi} \cos.\ (\theta - \theta') \ldots 10.2660$$

$$\log.\ (a) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ .8886$$

$$1.1546 \ldots \text{No.} = 14.275,$$

$$\log.\ \frac{\tan.\ \phi}{\tan.\ \phi'}\ .\cos.\ (\theta - \theta') \ldots\ldots\ldots\ldots\ 9.7096$$

$$\log.\ (b) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ 1.2528$$

$$.9624 \ldots \text{No.} = 9.1707\ ;$$

$$\therefore\ \delta\theta = -(7''.737 - 14''.275) = +\ 6''.53,$$

$$\delta\theta' = -(17''.898 - 9''.1707) = -\ 8''.727.$$

The *progression*, therefore, of Jupiter's node, for 100 years, is about 653″, the regression of Saturn's, for the same period, 873″: supposing, which is a material circumstance, during that period, the plane of the ecliptic for 1800 to remain fixed. The plane of the *true* ecliptic must, however, from the principles we are reasoning on, be perpetually oscillating.

The *whole* mean annual progressions and regressions of the nodes of the orbits of Jupiter and Saturn, on the plane of a fixed ecliptic, differ, very little, from those that have been just assigned, and which are derived from their mutual perturbation. The effect of the other planets is small except with reference to the variable or true ecliptic: and, in that case, their effect on the nodes of Jupiter and Saturn is an indirect one.

The secular motions of the nodes of Jupiter and Saturn on the planes of each other's orbits are uniform: but, with regard to a fixed plane, variable, and by reason (see p. 437.) of the second terms which involve

$$\frac{\tan.\ \phi'}{\tan.\ \phi}\cos.\ (\theta'-\theta),\quad \frac{\tan.\ \phi}{\tan.\ \phi'}\cos.\ (\theta'-\theta).$$

These quantities vary from the variability of ϕ', ϕ, θ', and θ. If ϕ', as we have seen (p. 435,) decreases, ϕ will simultaneously increase: when ϕ' is at its lowest value, ϕ will be at its greatest: and if $\frac{\tan.\ \phi'}{\tan.\ \phi}\cos.\ (\theta'-\theta)$, should, by the diminution of ϕ' and the corresponding augmentation of ϕ, become less than 1, Jupiter's nodes would *regress*, but (see p. 437.) Saturn's nodes would not necessarily *progress:* whether they did or not, must depend on the corresponding values of ϕ', ϕ, θ', and θ *.

ϕ' represents the inclination of the orbit of the disturbing planet: if that quantity should be less than ϕ, the nodes of the disturbed planet *must* regress on the plane of the ecliptic: if ϕ' should be greater than ϕ the nodes *may* progress. Now of all the orbits, Mercury's has the greatest inclination ($\phi' = 7^0$), and if we examine the Table of the Elements (see *Astron.* 286.) for the several values of $\theta' - \theta$, we shall find that in every case (excepting the newly discovered minute planets),

$$\frac{\tan.\ 7^0}{\tan.\ \phi}\cos.\ (\theta'-\theta) > 1:$$

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* The matter must be determined by computing $\frac{\tan.\ \phi}{\tan.\ \phi'}\cos.\ (\theta'-\theta)$. If, with Lagrange, (see *Berlin Acts,* 1782, pp. 249, 250.) we take $47'$, $2^0\ 32'\ 40''$ to be least and greatest inclinations of Saturn's orbit: $2^0\ 2'\ 30''$, $1^0\ 17'\ 10''$ the corresponding greatest and least inclinations of Jupiter's orbit, we have, as an extreme case, $\frac{\tan.\ \phi}{\tan.\ \phi'} = \frac{\tan.\ 2^0\ 2'\ 30''}{\tan.\ 47'}$; in which case, if $\cos.\ (\theta'-\theta)$ were not less than $\frac{2}{5}$, $\frac{\tan.\ \phi}{\tan.\ \phi'}\cos.\ (\theta'-\theta)$ would be > 1, and Saturn's node would *progress.*

consequently, the disturbing force of Mercury, if that were the sole disturbing force, would, at the present epoch, render the nodes of every planet *progressive.*

The motion of the Moon's nodes may be deduced from the preceding formulæ: and in this case, from the minuteness of $\frac{r}{r'}$, R admits of a very easy development: for, R, excluding the terms that do not involve θ and ϕ, equals (see p. 408.)

$$2\,m'\,r\,.\,\sin.^2\frac{\phi}{2}\ \sin.\,v'\,.\,\sin.\,v\left(\frac{r'}{[r'^2-2rr'\cos.(v'-v)+r^2]^{\frac{3}{2}}}-\frac{1}{r'^2}\right),$$

which, developed, equals

$$\frac{3\,m'\,r^2}{r'^3}\,.\,\sin^2.\,\frac{\phi}{2}\,.\left\{\begin{matrix}\frac{1}{2}+\frac{1}{2}\cos.\,(2\,v'-2\,v)\\ -\frac{1}{2}\cos.\,2v'-\frac{1}{2}\cos.\,2\,v\end{matrix}\right\}$$

v' and v being reckoned from the intersection of the planes (see p. 407.).

If the angles v, v', are to be reckoned from points distant in their respective orbits from the above intersection by the quantity θ, (θ being, in fact, the longitude of the node to be reckoned on each orbit), then the three cosines of the preceding expression will become

$$\cos.\,(2\,v'-2\,v),\quad \cos.(2v'-2\,\theta),\quad \cos.\,(2\,v-2\theta).$$

The constant part in the preceding expression, and on which the secular motion of the nodes depends, is

$$\frac{3\,m'\,r^2}{2\,r'^3}\sin^2.\,\frac{\phi}{2},\ \text{or}\ 3\,m'\,\frac{a^2}{a'^3}\ \sin^2.\,\frac{\phi}{2}:$$

this (see p. 416.) answers to F: consequently,

$$\frac{\delta\,\theta}{d\,t}=-\frac{1}{\sqrt{a}\,.\sin.\,\phi}\cdot\frac{3\,m'\,a^2}{2\,a'^3}\times\sin.\,\frac{\phi}{2}\cos.\,\frac{\phi}{2}$$

$$=-\frac{3\,m'}{4}\cdot\frac{a^3}{a'^3}\,n,$$

the value of $\delta\,\theta$ in one year, or,

$$\int \delta\theta = -3m'\frac{a^3}{a'^3}.nt^*,$$

nearly equals (see pp. 200, &c.) 20° 7′.

In order to determine the real position of the node at any assigned time, account must be made of the three cosines in the preceding expression, and $\frac{dR}{d\phi}$ must be accordingly computed: for the purpose of such computation R, and thence $\frac{\delta\theta}{dt}$, is conveniently expressed. But, for the purpose of illustration, and for shewing, as Newton has done, Prop. XXI. Lib. 3. in what positions of the Moon and the nodes of her orbit the latter are progressive, and in what regressive, this expression, namely,

$$\frac{\delta\theta}{dt} = -3m'\frac{a^3}{a'^3}n[\sin.(v-\theta)\cos.(v-v')\sin.(v'-\theta)],$$

is most convenient; which expression is, by means of known Trigonometrical formulæ, easily deducible from the former, and is, in fact, Newton's expression †.

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* The general expression for the secular regression of the node of any planet on the orbit of the disturbing planet is (see p. 438.)

$$-\frac{m'n}{4}.\frac{a^2}{a'^2}.B'a'^3 \times t,$$

in which B' is the coefficient of the second term of the development of $(a'^2 - 2aa'\cos.\omega + a^2)^{-\frac{3}{2}}$: now make $a = 1$, $nt = 360°$, (t then being the periodic time), and the regression equals

$$-m'B'a'.90°,$$

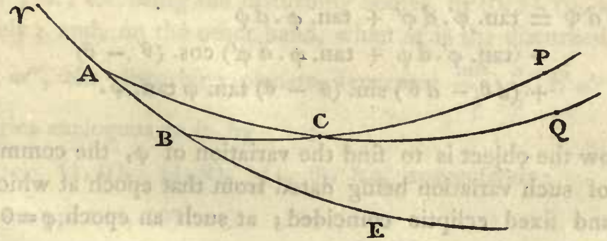
which is exactly the expression which Lalande gave in the *Mem. Acad. des Sciences*, 1758, p. 252. It is a little remarkable that authors, who have written subsequently to Lalande, should not have adopted this simple expression. The only difficulty attending its use is in the computation of B': 'la recherche,' says the Author, 'en est souvent tres difficile.' The difficulty, however, is completely removed by Chap. XVIII. of this Work.

† 'Est igitur velocitas nodorum, &c. ut contentum sub sinubus trium angulorum, &c. Prop. XXX. Lib. 3.

We have already found (see p. 424.) the most interesting instance, for exemplifying and illustrating the formula for the variation of the eccentricity, in the solar orbit. We may resort, with like success, to the same instance for the means of illustrating the expressions of the variations of the node and inclination.

The plane of the ecliptic, which is the plane of the Earth's orbit, must, like the plane of the orbit of any other planet, be made to oscillate by planetary perturbation.

If we consider, as a fixed ecliptic, that in which the Earth's orbit is at any particular epoch (1750 for instance) then, at another epoch, the Earth will be found in a different ecliptic; that in which the orbit is at any assigned time, is called the *true* Ecliptic. It is this ecliptic to which we really refer the heavenly bodies; from which we measure their latitudes, and along which we measure their longitudes. The expressions therefore for the variations of the inclinations and nodes (see pp. 419, &c.) require some alteration in order to be adapted to Astronomical usage: for they refer to a fixed plane. In order to adapt these expressions to the variable plane of the true ecliptic: let ΥABE represent the



fixed ecliptic, ACQ the orbit of the Earth, or the true ecliptic, BCP the orbit of any other planet m'.

Let

l be its longitude from Υ,
ϕ' its inclination (CBE) with the orbit ABE,
Φ its inclination (PCQ) with the orbit ACQ,
Θ the longitude of C measured on the orbit QCA,
θ and θ' the longitudes of A, and B;

then, the inclinations ϕ, ϕ', &c. being supposed to be very small, the latitude of m' (at the point P) with reference to the orbit QCA, is very nearly equal to its latitude with reference to the orbit ABE minus the latitude of m' from the same orbit, supposing m' having still its longitude l, to be placed on the orbit ACQ; but by Naper's Rule (see *Trig.* p. 136.)

tan. lat. = tan. inclination × sin. dist. from node,

and, accordingly, assuming the latitudes (which are small angles) instead of their tangents, we have, very nearly,

$$\begin{aligned}\tan.\,\Phi \times \sin.\,(l - \Theta) = {} & \tan.\,\phi'\cdot\sin.\,(l - \theta') \\ & - \tan.\,\phi\,.\sin.\,(l - \theta),\end{aligned}$$

and, by equating, respectively, the terms which contain sin. l, and those which contain cos. l.

$$\tan.\,\Phi\cos.\,\Theta = \tan.\,\phi'.\cos.\,\theta' - \tan.\,\phi\cos.\,\theta,$$
$$\tan.\,\Phi\sin.\,\Theta = \tan.\,\phi'.\sin.\,\theta' - \tan.\,\phi\sin.\,\theta.$$

If we add the square of the upper line to the square of the lower, we shall have $\tan^2.\,\Phi$; and $\tan.\,\Theta$, by dividing the lower by the first. From the former value, we have, nearly, (Φ, ϕ, ϕ', being all very small)

$$\begin{aligned}\tan.\,\Phi\, d\,\Phi = {} & \tan.\,\phi'.d\,\phi' + \tan.\,\phi\,.d\,\phi \\ & - (\tan.\,\phi'\cdot d\,\phi + \tan.\,\phi\,.d\,\phi')\cos.\,(\theta' - \theta) \\ & + (d\,\theta - d\,\theta')\sin.\,(\theta' - \theta)\tan.\,\phi\tan.\,\phi'.\end{aligned}$$

Now the object is to find the variation of ϕ, the commencement of such variation being dated from that epoch at which the true and fixed ecliptic coincided; at such an epoch $\phi = 0$, and $\tan.\,\Phi = \tan.\,\phi'$: accordingly,

$$\begin{aligned}d\Phi, \text{ or } \delta\,\Phi = {} & \delta\,\phi' - \delta\,\phi\,.\cos.\,(\theta' - \theta) \\ & + (\delta\,\theta' - \delta\,\theta)\sin.\,(\theta' - \theta)\tan.\,\phi\,^*.\end{aligned}$$

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* We cannot erase this term, because $\delta\,\theta$ (see p. 437.) contains in its expression fractions such as

$$\frac{\tan.\,\phi'}{\tan.\,\phi},\quad \frac{\tan.\,\phi''}{\tan.\,\phi}.$$

If, by a like process, we find $\delta\,\Theta$, we shall have

$$\delta\,\Theta = \delta\,\theta' - \delta\,\theta \, . \frac{\tan.\,\phi}{\tan.\,\phi'} \cos.\,(\theta - \theta')$$

$$+ \frac{\delta\,\phi}{\tan.\,\phi'} \sin.\,(\theta' - \theta).$$

It now remains to substitute from pp. 437, &c. the values of $\delta\,\phi$, $\delta\,\phi'$, $\delta\,\theta$, $\delta\,\theta'$: but, since in this enquiry it is necessary to consider the perturbations of more than two planets, it will be found *convenient* to use symbols such as are described in the note of p. 421. Let then m be supposed the Earth, m', m'', m''', &c. Mercury, Venus, Mars, &c. Compute, by Chapter XVIII. B' and the quantities that are analogous to it : and, m, or the Earth, being the disturbed, and m', or Mercury, the disturbing planet; represent

$$\frac{m' n}{4} B' a'^3 . \frac{a^2}{a'^2} \text{ by } (0, 1).$$

Again, m'' being the disturbing planet, represent the above quantity by

$$(0, 2),$$

and, m''', m^{iv}, &c. being the disturbing bodies, by (0, 3), (0, 4), repectively ; and, on the other hand, when m' is the disturbed, and m, m'', m''', the disturbing planets, represent $\frac{mn'}{4} . \frac{a}{a'} B' a'^3$ and quantities analogous to it, by

$$(1, 0), \quad (1, 2), \quad (1, 3), \text{ \&c. respectively;}$$

then, see pp. 434, 437.

$$\frac{\delta\,\phi}{d\,t} = (0, 1) \tan.\,\phi' \sin.\,(\theta - \theta') + (0, 2) \tan.\,\phi'' \sin.\,(\theta - \theta'') + \text{\&c.}$$

$$\frac{\delta\,\phi'}{d\,t} = (1, 0) \tan.\,\phi \sin.\,(\theta' - \theta) + (1, 2) \tan.\,\phi'' \sin.\,(\theta' - \theta'') + \text{\&c.}$$

$$\frac{\delta\,\theta}{d\,t} = -\,[(0, 1) + (0, 2) + (0, 3)]$$

$$+ (0, 1) . \frac{\tan.\,\phi'}{\tan.\,\phi} \cos.\,(\theta - \theta') + (0, 2) \frac{\tan.\,\phi''}{\tan.\,\phi} \cos.\,(\theta - \theta'') + \text{\&c.}$$

$$\frac{\delta\theta'}{dt} = -[(1, 0) + (1, 2) + (1, 3)]$$

$$+ (1, 0)\,\frac{\tan.\phi}{\tan.\phi'}\cos.(\theta' - \theta) + (1, 2)\,\frac{\tan.\phi''}{\tan.\phi'}\cos.(\theta' - \theta'') + \&c.$$

If we now substitute these values in the expressions for $\frac{\delta\Phi}{dt}$, $\frac{\delta\Theta}{dt}$, we have, making $\tan.\phi$, wherever it occurs, $= 0$,

$$\frac{\delta\Phi}{dt} = [(1, 2) - (0, 2)]\tan.\phi''\sin.(\theta' - \theta'')$$

$$+ [(1, 3) - (0, 3)]\tan.\phi'''\sin.(\theta' - \theta''')$$

$$+ \&c.$$

and

$$\frac{\delta\Theta}{dt} = -[(1, 0) + (1, 2) + (1, 3)] - (0, 1)$$

$$+ [(1, 2) - (0, 2)]\frac{\tan.\phi''}{\tan.\phi'}\cos.(\theta' - \theta'')$$

$$+ [(1, 3) - (0, 3)].\frac{\tan.\phi'''}{\tan.\phi'}\cos.(\theta' - \theta''')$$

$$+ \&c.$$

These expressions, as it has been said, are convenient for Astronomical uses, since they determine the variation of the inclination of a planet's orbit to the *true* ecliptic, and the regression of its node on the same ecliptic; and it is from such expressions that the Tables of the variations of nodes and inclinations are constructed.

The formula $\frac{\delta\Phi}{dt}$, which expresses the variation of the inclination of any orbit (Jupiter's for instance) to the true ecliptic, includes, besides the mutual perturbation of the Earth and Jupiter, the effect of the perturbations of the other planets. From such effect arises the deviation of the plane of the true ecliptic from the plane of that ecliptic which is considered as a fixed plane, and in which, at a particular epoch, the Earth's orbit was found. The *obliquity of the ecliptic* is the technical denomination of the inclination of the plane of that circle to the plane of the equator. There must, therefore, by reason of the deviation of the plane of the *true* from that of the *fixed* ecliptic, arise a change in the *obliquity*, or a variation of that inclination, which, at the epoch referred to, subsisted between the planes of the equator and of the *fixed* ecliptic. The *diminution*, then, (for such it is) *of the obliquity of the ecliptic*, arises from the disturbing forces of the planets, and may easily be investigated by means of the preceding formulæ.

Let ABE (fig. of p. 443.) represent the equator, PCB the fixed ecliptic, QCA the true ecliptic ; B will be the intersection of the fixed ecliptic and equator, A of the true ecliptic and equator, and BA, will, accordingly, represent that *displacement* of the equinoctial points which arises from the inclination ($\angle\, PCQ$) of the true and fixed ecliptic ; and, since longitudes are measured along the ecliptic, $CA - CB$ will represent the error or deviation of the longitude of the equinoctial points due to the above inclination PCQ, and AB is the corresponding deviation in right ascension arising from the same cause.

Let $\angle\, PCQ = \phi,$

$$\angle\, CBE = E, \quad \angle\, CAB = E - \Delta E,$$

$$CB = \lambda, \quad CA = \lambda + \Delta\lambda :$$

$$AB = a,$$

then we have, (see *Trig*. ed. 2. Chap. ix.)

$$\cos. (E - \Delta E) = \cos. \phi \,.\, \cos. E + \sin. \phi \, \sin. E \, \cos. \lambda,$$

whence, by expanding, &c. we have, very nearly, (since ΔE, ϕ, are very small),

$$\sin. E \,.\, \Delta E = \sin. \phi \, \sin. E \, \cos. \lambda,$$

$$\text{or } \Delta E = \phi \cos. \lambda, \text{ or, nearly, } = \tan. \phi \cos. \lambda,$$

Again, (see *Trig*. p. 131.)

$$\frac{\sin. (E - \Delta E)}{\sin. \phi} = \frac{\sin. \lambda}{\sin. a};$$

$$\therefore a = \phi \,.\, \frac{\sin. \lambda}{\sin. E}, \text{ or, nearly, } = \frac{\tan. \phi \sin. \lambda}{\sin. E}.$$

Lastly,

$$\frac{\sin.(\lambda + \Delta\lambda)}{\sin.\lambda} = \frac{\sin. E}{\sin.(E-\Delta E)};$$

$$\therefore \Delta\lambda = \frac{\tan.\lambda}{\tan. E}$$

$$= \frac{\tan.\phi \sin.\lambda}{\tan. E}$$

$$\text{or} = \phi \sin.\lambda \text{ cotan. } E,$$

ϕ, therefore, being the whole deviation of the true ecliptic from a particular epoch, $\Delta E\ (= \phi \cos.\lambda)$ will be the corresponding change of obliquity, and the annual diminution of obliquity will be

$$\frac{\delta\phi \cos.\lambda - \phi.\delta\lambda.\sin.\lambda}{dt},$$

$$\text{or, } \frac{\delta\phi}{dt}\cos.\lambda - \frac{\delta\lambda}{dt}\tan.\phi.\sin.\lambda.$$

Now λ is the distance of the point of intersection of the true and fixed ecliptic from the intersection of the equator and ecliptic: it is, therefore, the longitude of the node of the true ecliptic on the fixed ecliptic and corresponds (see p. 443.) to θ: accordingly, we have

$$\frac{\delta\phi}{dt}\cos.\lambda = (0,1)\tan.\phi' \sin.(\lambda - \theta')\cos.\lambda$$
$$+ (0,2)\tan.\phi'' \sin.(\lambda - \theta'')\cos.\lambda + \&c.$$

$$\frac{d\lambda}{dt}\tan.\phi \sin.\lambda = -[(0,1) + (0,2) + (0,3)]\tan.\phi \sin.\lambda$$
$$+ (0,1)\tan.\phi' \cos.(\lambda - \theta')\sin.\lambda$$
$$+ (0,2)\tan.\phi'' \cos.(\lambda - \theta'')\sin.\lambda$$
$$+ \&c.$$

Hence, making tan. $\phi = 0$, (which it is at the commencement of the epoch), we have

$$\frac{\delta\phi}{dt}\cos.\lambda - \frac{\delta\lambda}{dt}\tan.\phi \sin.\lambda =$$

[*] $$-(0,1)\,\text{tan.}\,\phi'\,\text{sin.}\,\theta' - (0,2)\,\text{tan.}\,\phi''\,\text{sin.}\,\theta'' - (0,3)\,\text{tan.}\,\phi'''\,\text{sin.}\,\theta''' - \&\text{c.}$$

to represent the mean annual diminution of the obliquity; such diminution being reckoned from that epoch at which the true and fixed ecliptic coincided.

The whole motion in longitude ($\Delta\lambda$) of the equinoxes corresponding to the angle of deviation ϕ is (see p. 448.) tan. ϕ sin. λ co-tan. E; therefore the annual motion is, nearly,

$$\text{co-tan.}\,E\left(\frac{\delta\phi}{dt}\,\text{sin.}\,\lambda + \frac{\delta\lambda}{dt}\,\text{cos.}\,\lambda\,\text{tan.}\,\phi\right),$$

which (since, as before, $\lambda = \theta$) is, by the formulæ of p. 448.

$$\text{cot. obl}^{y}.\ [(0,1)\,\text{tan.}\,\phi'\,\text{cos.}\,\theta' + (0,2)\,\text{tan.}\,\phi''\,\text{cos.}\,\theta'' + \&\text{c.}]$$

The annual motion of the equinoxes in right-ascension, or $\frac{\delta a}{dt}$ is (see p. 447.)

$$\text{co-sec. obl}^{y}.\ [(0,1)\,\text{tan.}\,\phi'\cdot\text{cos.}\,\theta' + (0,2)\,\text{tan.}\,\phi''\,\text{cos.}\,\theta'' + \&\text{c.}]$$

These are the variations of the obliquity of the ecliptic, and of those motions of the equinoctial points which arise from the disturbing forces of the planets, and which are independent of that inequality which is technically called the *Precession* of the *Equinoxes* (see *Astron.* Chap. XIV).

In order to deduce the arithmetical values of the preceding formulæ we must previously deduce those of (0, 1), (0, 2), &c.

Now, (see p. 445.) (0, 1) represents the value of $\frac{m'na^2}{4a'^2}\cdot B'a'^3$, when m is the disturbed, m' the disturbing body, and a, a', are their respective mean distances. The term $B'a'^3$, is, according to the value of $\frac{a}{a'}$, to be computed by the methods of Chap. XVIII.

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[*] This expression agrees with that which Lagrange has given in the *Berlin Memoirs* for 1782, p. 209.

Hence, when Mercury (m') is the disturbing body,

$$(0,\ 1) = .09757,$$

and, since

$\phi' = 6^\circ\ 0'\ 55''$*, $\theta' = 45^\circ\ 57'\ 25''$, $(0,\ 1)\ \tan.\phi'.\sin.\theta' = .008521$,

when Venus (m'') is the disturbing body,

$$(0,\ 2) = 5.427,$$

and, since

$\phi'' = 3^\circ\ 23'\ 34''$, $\theta'' = 74^\circ\ 52'\ 53''$, $(0,\ 2)\ \tan.\phi''\ \sin.\theta'' = .309950$,

when Mars (m''') is the disturbing body,

$$(0,\ 3) = .43299$$

and, since

$\phi''' = 1^\circ\ 51'\ 4''$, $\theta''' = 48^\circ\ 14'\ 57''$, $(0,\ 3)\ \tan.\phi'''\ \sin.\theta''' = .010336$,

when Jupiter (m^{iv}) is the disturbing body,

$$(0,\ 4) = 6.9478,$$

and, since

$\phi^{\text{iv}} = 1^\circ\ 19'$, $\theta^{\text{iv}} = 98^\circ\ 25'\ 47''$, $(0,\ 4)\ \tan.\phi^{\text{iv}}\ \sin.\theta^{\text{iv}} = .158234$,

when Saturn (m^{v}) is the disturbing body,

$$(0,\ 5) = .3404\ \dagger,$$

and since

$\phi^{\text{v}} = 2^\circ\ 29'\ 41''$, $\theta^{\text{v}} = 111^\circ\ 56'\ 18''$, $(0,\ 5)\ \tan.\phi^{\text{v}}\ \sin.\theta^{\text{v}} = .013821$,

and, if we collect the several values of $(0,\ 1)\ \tan.\ \phi'\ \sin.\ \theta'$, &c. we shall have the whole annual diminution of the obliquity equal to

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* The values of ϕ', ϕ'', &c. are those of the epoch of 1750.

† The values of (0, 1) (0, 2), &c. are, see p. 445. the several values of $\frac{m'\,n}{4}\cdot\frac{a^2}{a'^2}\,B'\,a'^3$: but that quantity (see p. 438.) expresses the mean annual regression of the node of the orbit of m on the orbit of m', consequently, the preceding numerical values of (0, 1), (0, 2), &c. express the mean annual regressions of the nodes of the ecliptic on the respective orbits of Mercury, Venus, &c.: which regressions are (see the text,) nearly, 0′.097, 5″.43, 4″.33, 6″.947, 0″.34.

$$0''.500862,$$

and, accordingly, the *secular* diminution (meaning by that term the diminution in 100 years), will be

$$50''.0862.$$

We may consider then 50″ nearly to represent the secular diminution of the obliquity: which agrees tolerably well with observation *. It cannot be expected to agree with great exactness, since, on this head, there is, as we have already mentioned, some uncertainty. The diminution of the obliquity arises from the disturbing forces of the planets; the disturbing forces depend, in part, on the masses, and the masses of all the planets are not well ascertained. Venus is in this predicament: but, as it appears from the preceding computation, the effect of Venus, (on the assumption, indeed, of a conjectural but very probable value of her mass) is, in diminishing the obliquity, very nearly double that of any other planet. The mass of Venus, therefore, requires to be most accurately known in order to compute with accuracy the diminution of the obliquity; and contrariwise, the diminution nicely determined by observation is the fittest inequality for determining the mass of Venus: the *mass* and the *diminution* as objects of computation, are *implicitly* involved. The diminution (a very small quantity even in an hundred years) as a result of observation is not well known by reason of the inaccuracy of antient observations.

Still, however, the observations are sufficiently accurate to establish, beyond a doubt, the fact of a diminution of the obliquity: and the no great discrepancy between the results of observation and calculation renders it, at the least, probable that it is caused by the disturbing forces of the planets: that is, by the particles of their masses attracting the Earth with forces proportional to their number, and according to the law of the inverse square of the distance.

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* In 1750, according to Bradley and Lacaille, the mean obliquity was 23° 28′ 19″: in 1800, according to Maskelyne, Piazzi, and Delambre, 23° 27′ 57″. In 1813, by the new circle at Greenwich, it was, according to Mr. Pond, 23° 27′ 50″: the two first compared together give 44″: the first and last 46″ for the diminution of the obliquity in a century.

But of the three effects (see p. 449.) which the disturbing forces of the planets ought, on Newton's Principles, to produce on the plane of the Earth's orbit, the diminution of the obliquity is the only one which observation has hitherto been able to ascertain. That has been effected by observations of the Sun at the solstices, and of the latitudes of Stars situated near the solstices*. But the motions of the equinoctial points in longitude and right-ascension (see p. 449.) are too minute, and too blended with the inequality of the precession, to be separately exhibited. If we compute, according to the method of p. 450. the annual motions of the equinoctial points in longitude and right-ascension, they will be found respectively equal to 0″.1767, 0″.1926. Now these are, in their directions, opposite to the effects of *precession.* Whilst the latter increase the longitudes and right-ascensions of Stars, the former diminish them. If then we assume, as it is determined by the best observations, 50″.1 to be the mean annual precession, that quantity being the result of the action of the Sun and Moon (of the *Lunisolar* influence, as it is called) and of the perturbations of the planets, the effect of the latter in longitude, namely, 0″.1767 must be *added* to 50″.1, in order to expound the *Lunisolar precession* For, were it not for the *progression* in longitude of the equinoxes produced by the perturbation of the planets, the *Lunisolar precession,* would, by observation, appear to be larger: by just so much indeed as the *progression* diminishes it. The former, therefore, must be 50″.2767. And, in like manner, the precession in right-ascension common to all Stars (see *Astronomy,* p. 142.) due to the same cause must be

46″.1 + 0″.1926, or 46.2926.

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* The diminution of the obliquity, simply viewed as a phenomenon, may be accounted for either from the equator or the ecliptic changing its place. Tycho Brahé shewed that it was *truly* accounted for by the ecliptic changing its position: since the northern latitudes of Stars situated near the solstices were found to increase, and the southern to decrease.

Although, therefore, the *direct* movement of the equinoctial points arising from the displacing of the ecliptic, is, in observations, necessarily confounded with their retrograde movement arising from the displacing of the equator, yet, if we admit the preceding theory and results, we are enabled by them to assign what is separately due to the action of the Sun and Moon.

The direct movement of the equinoctial points arising from the displacing of the ecliptic lessens the longitude of heavenly bodies: the precession increases them. The Sun, therefore, by reason of the former, after quitting the equinox, returns later to the same, and sooner by reason of the latter. The former prolongs the tropical year, the latter shortens it; considering, for a moment, the just value of the tropical year to be that which it would have, were neither the equator nor the ecliptic displaced. If, however, the direct and retrograde motions of the equinoctial points were always the same, the *true* tropical year (that which really takes place) would always be of the same length. It would be of the same length now, as it was at the time of Hipparchus. But the fact is otherwise. Since his time, both the precession and the *progression* of the equinoxes caused by the displacing of the ecliptic have varied, and not by equal degrees.

That the latter has varied may easily be inferred from its expression (p. 449.); the obliquity, the inclinations and longitudes of the nodes (the quantities ϕ', ϕ'', &c. θ', θ'', &c.) were, by reason of the disturbing forces, all of different values at the beginning of the Christian Æra from what they are at present. From their present values, the motion of the equinoxes (see p. 452.) was found equal to $0''.1767$: and if, by the same formula of p. 449, we compute its value for the beginning of our Æra, it will be found to be about $0''.48$, and consequently, if the difference of the real precessions depended solely on those two quantities, the true precession in 1800 would be greater than the true precession in the year 1, by $0''.48 - 0''.1767$, or $0''.3033$: and, accordingly, the tropical year, at the former Æra, would be shorter than the tropical year at the commencement of the Æra, by as much time as the Sun would consume in describing $0''.303$ of longitude.

But another cause operates; the Lunisolar precession (that which is caused by the action of the Sun and Moon on the protuberant equatoreal parts of the Earth) varies as the cosine of the obliquity. The obliquity then decreasing, the precession must be increased, and it will now be about 0″.09 greater than it was at the commencement of our Æra: the true precession therefore of 1800 will now be greater than the true precession of the year 1 by 0″.303 + 0″.09, or, 0″.393. The Sun, (assuming its mean motion in 24 hours to be 59′ 8″), would describe this space (0″.393) in about 9″.3, which is the computed excess of the tropical year at the beginning of the Æra above the present tropical year.

It is the *computed* excess; being merely a result from theory. Antient observations are inaccurate far beyond 9 seconds, and, consequently, we can only say that the *progression* of the equinoctial points from the disturbing forces of the planets is a *probable* result. The point, however, may be settled in future times, if observations should then be made as accurately as they are at present*.

Besides the *progression* of the equinoctial points, there are other inequalities, discussed in this Chapter, that, at present, ought to be viewed as mere results of theory. Such are, for instance, the variations of the inclinations of the planes of the orbits of planets. These, hitherto, have not been determined by observation: they are too minute, and antient observations are too inaccurate; if the former are as minute as theory shews them to be, it is hopeless to expect to determine them by the latter. This is another point reserved for future Astronomers.

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* The matter can never be altogether free from uncertainty. If, by *observations,* made 500 years hence, compared with modern, the tropical year should then appear to be less, the fact might be accounted for by supposing the *Lunisolar* precession to be less diminished by the direct motion of the equinoctial points: *the mean quantity of the Lunisolar precession itself being always supposed the same.* And it would be accounted for, with a high degree of probability, if the *computed progression* (see pp. 452, &c.) agreed with the difference of the lengths of the tropical years as made out from the comparison of observations.

There is, however, one exception to what has been just said. The diminution of the obliquity, which is a consequence of a change of inclination in the Earth's orbit, may now be considered as established by observations, although 70 years ago there were Astronomers who asserted that it was constant, and, which is more strange, denied that it could vary on Newton's Principles.

We have in the present Chapter deduced and exemplified the expressions for the secular inequalities of the elements of a planet's orbit. We have also, on restricted conditions indeed, established some very curious properties concerning the limits within which both the variations of the eccentricities and the inclinations are confined. In such and like properties consists the *stability* of the Planetary System: which, of all the results furnished us by *Physical Astronomy*, is, perhaps, the most interesting. It merits then some farther consideration; and, in the next Chapter, we will endeavour to render more general those which are to be considered (see pp. 422, 435.) as its essential theorems.

CHAP. XXIII.

Stability of the Planetary System with regard to the Mean Distances. The Mean Distances subject only to Periodical Inequalities and not to Secular. Stability of the Planetary System with regard to the Eccentricities and Inclinations. Theorems which express the Conditions to which their Variations are subject.

THE constant parts of the development of R, (so it appears by p. 414,) do not contain the quantity ϵ: and since

$$\delta a = -\frac{2}{\mu} a^2 . d R = -\frac{2}{\mu} a^2 . \frac{d R}{d \epsilon} n d t,$$

it was thence inferred that, with regard to such constant parts, $\frac{d R}{d \epsilon}$ was $= 0$: in other words, that the axis major was subject to no secular variation.

This, which is an important point, may be considered under another point of view.

The arguments of terms in the value of R (see pp. 279, 281, &c.) independent of the eccentricities, are

$$p(n' t - n t + \epsilon' - \epsilon);$$

of terms involving the first powers of the eccentricities, the arguments are

$$p(n' t - n t + \epsilon' - \epsilon) + n t + \epsilon - \pi,$$

$$\text{and} \quad p(n' t - n t + \epsilon' - \epsilon) + n' t + \epsilon' - \pi',$$

that is,

$$p n' t - (p - 1) n t + p \epsilon' - (p - 1) \epsilon - \pi,$$

$$\text{and} \quad (p + 1) n' t - p n t + (p + 1) \epsilon' - p \epsilon - \pi'.$$

The arguments of terms involving the squares of the eccentricities will be

$$p n' t - (p - 2) n t + \&c.$$
$$\text{and} \quad (p + 2) n' t - p n t + \&c.$$

so that, it is plain, we may generally represent a term in the development of R, by

$$P . \cos. (p' n' t - p n t + A),$$

in which p', p, will be integers having their difference $\pm (p' - p)$, connected with the powers or products of the eccentricities that are involved in the coefficient P.

Now dR is the differential of R, when those quantities are made to vary which determine the place of the body m (be they co-ordinates, or radius vector and longitude): but these quantities being expressed, by means of the variable quantity $n t$ and of certain constant quantities, the differential of R corresponding to the term

$$P . \cos. (p' n' t - p n t + A),$$

must be obtained by making $n t$ vary; accordingly,

$$d R = P p n d t . \sin. (p' n' t - p n t + A),$$

and

$$\delta a = - \frac{2}{\mu} a^2 . P p n d t . \sin. (p' n' t - p n t + A).$$

Now p', p are integers, and $p' - p$ may $= 0$, or ± 1, or ± 2, or &c. and if p' and p could be taken such that

$$p' n' - p n = 0,$$

then there would result, in the above expression, at least one term in the variation of a equal to

$$- \frac{2}{\mu} a^2 . P p n d t . \sin. A,$$

A being constant; and, accordingly, there would result in

$$- \int \frac{\delta a}{a^2} = \frac{1}{a},$$

a term $\frac{2}{\mu} P p n t . \sin . A$ increasing with the time, altering and continuing to alter the mean distance. But, so it happens, the mean motions, n and n', of the disturbed and disturbing planets are such that $p' n'$ can never equal $p n$. If the Earth be the planet disturbed by the actions of all the others, its mean motion (see Table of Periods, *Astronomy*, p. 283.) is not commensurable with the mean motion of any other planet. Its mean distance, therefore, suffers no *secular* change from the disturbing forces of the planets. The same holds good of the mean distance of every other planet, and for the same reason. The mean motion of Jupiter, for instance, is not to the mean motion of any other planet as number is to number. Twice Jupiter's mean motion is indeed, as we have seen in Chapter XIX, nearly equal to five times Saturn's; the consequence of which is, that their motions are affected with inequalities of a very long period: so long, indeed, that the inequalities are of the nature of secular inequalities, and become blended with the mean motions; and this latter is a result deducible from the preceding expression: for, make $p' = 5$, and $p = 2$, and then

$$- \frac{\delta a}{a^2} = \frac{4}{\mu} P n d t \sin . (5 n' t - 2 n t + A),$$

which expression will, for a great length of time, continue of the same sign; since, $5 n' - 2 n$ being very small, t must be very great before $5 n' t - 2 n t$ from 0 can become 180°. But, the mean distance continuing either to increase or decrease during a long period, the mean motion will continue to decrease or to increase during the same period.

By whatever method, then, we examine the effect of the disturbing forces of the system on the mean distances of the planets, it appears that those distances are subject to no secular change. They vary *periodically*, that is, they increase for a time by small quantities, and again, having reached a certain limit, by like degrees decrease: for, such is the nature of the change indicated by the term

$$- \frac{2 a^2}{\mu} P p n d t . \sin . (p' n' t - p n t + A).$$

From the invariability of the mean distances of the planets, we will proceed to consider another main point in the stability of the planetary system, and which consists in the restriction of the variations of the eccentricities within certain, and those very small, limits.

By p. 417, (neglecting the squares of the eccentricities),

$$\delta e = \frac{a}{e} \cdot \frac{dF}{d\pi} \cdot n\,dt,$$

in which, since the first, second, and fourth terms of the value of F (see p. 414.) do not involve π, we may suppose F represented by this restricted value, namely, $\frac{m' C'}{4} a a' . e e' . \cos.(\pi' - \pi)$.

This is the expression if m' be the sole disturbing body; but introduce a second and a third body, &c. m'', m''', &c. and the value of F will be increased by two terms similar to

$$\frac{m' C'}{4} a a' e e' \cos.(\pi' - \pi),$$

namely,

$$\frac{m''}{4} C''. a a'' . e e''. \cos.(\pi'' - \pi), \quad \frac{m'''}{4} C'''. a a''. e e'''. \cos.(\pi''' - \pi).$$

Now,

$$n = \frac{1}{a^{\frac{3}{2}}}, \quad a n = \frac{1}{\sqrt{a}},$$

and accordingly,

$$m \sqrt{a} . e \delta e = m \frac{dF}{d\pi} dt,$$

and similarly,

$$m' \sqrt{a'} . e' \delta e' = m' \frac{dF}{d\pi'} dt,$$

$$m'' \sqrt{a''} . e'' \delta e'' = m'' \frac{dF}{d\pi''} dt;$$

&c.

but, as it is plain from the value of F, (p. 459, l. 10.)

$$m.\frac{dF}{d\pi} + m'.\frac{dF}{d\pi'} + m''.\frac{dF}{d\pi''} + \&c. = 0,$$

consequently,

$$m\sqrt{a}.e\,\delta e + m'\sqrt{a'}.e'\,\delta e' + m''\sqrt{a''}.e''\,\delta e'' + \&c. = 0,$$

whence,

$$m\sqrt{a}.e^2 + m'\sqrt{a'}.e'^2 + m''\sqrt{a''}.e''^2 + \&c. = K,$$

in which equation the correction K is a constant quantity.

Now K is to be computed, and the rest of the process conducted exactly as it was in p. 422, and as in that, so in the present case, when account is made of the disturbing forces of all the planets, K (since e, e', e'', &c. are all very small) will be a small quantity. But K being a small quantity, $m\sqrt{a}.e^2$, $m'\sqrt{a'}.e'^2$, &c. must each, at the least, be less than K, or, e^2, e'^2, e''^2, &c. must each, at the least, be less than $\frac{K}{m\sqrt{a}}$, $\frac{K}{m'\sqrt{a'}}$, $\frac{K}{m''\sqrt{a''}}$, &c. so that, as when two bodies only were considered (see p. 423.) the respective augmentations of e, e', e'', &c. will be confined within very narrow limits. This then is the second point in the stability of the planetary system. The eccentricities vary indeed from disturbing forces, but they alternately increase and decrease. The orbits may be said to oscillate about a mean state of *ellipticity*, whilst their major axes remain invariable. The conditions to which their eccentricities are subjected, is expressed by the theorem of l. 6.

The third point of the *stability* of the planetary system consists in the oscillations of the inclinations of the orbits of planets about a *mean* state of inclination; which may be thus proved.

By the expressions of pp. 418, 419,

$$\delta\phi = \frac{1}{h.\gamma}.\frac{dF}{d\theta}dt,$$

$$\delta\theta = -\frac{1}{h\gamma}.\frac{dF}{d\phi}dt,$$

in which F* (see p. 416.) may be represented by the last term of its value, namely,

$$\frac{m'\,B'}{2}\,.\,a\,a'.\sin^2.\,\frac{I}{2}\,,\quad\text{or}\quad \frac{m'\,B'}{4}\,a\,a'.\,(1-\cos.\,I),$$

which is equal (p. 418.)

$$\frac{m'\,B'}{4}\,a\,a'\,[1-\cos.\,\phi\,\cos.\,\phi'-\sin.\,\phi\,\sin.\,\phi'.\cos.\,(\theta-\theta')],$$

and, as in the former case, if besides m', other disturbing bodies as m'', m''', &c. act, F will be augmented by terms analogous to the preceding.

Now, since

$$\frac{d\,F}{d\,\theta}=\frac{m'\,B'}{4}\,a\,a'.\sin.\,\phi\,\sin.\,\phi'\,\sin.\,(\theta-\theta')$$
$$+\,\frac{m''}{4}\,B''\,a\,a''.\sin.\,\phi\,\sin.\,\phi''\,\sin.\,(\theta-\theta''),$$
$$+\,\&c.$$

$$\text{and}\quad \frac{d\,F}{d\theta'}=-\,\frac{m'\,B'}{4}\,a\,a'\,\sin.\,\phi\,\sin.\,\phi'\,\sin.\,(\theta-\theta')$$
$$-\,\frac{m''\,B''}{4}\,a\,a''\,\sin.\,\phi\,\sin.\,\phi''\,\sin.\,(\theta-\theta'')$$
$$-\,\&c.$$

it is plain, that

$$m.\frac{d\,F}{d\,\theta}+m'\frac{d\,F}{d\,\theta'}+m''\,\frac{d\,F}{d\,\theta''}+\&c.=0,$$

consequently,

$$m\,h\,\gamma\,.\,\delta\,\phi+m'\,h'\,\gamma'.\,\delta\,\phi'+m''\,h''\gamma''.\,\delta\,\phi''+\&c.=0,$$

or, since $h=\surd a$, $h'=\surd a'$, &c. very nearly,

and $\gamma=\tan.\,\phi=\phi$, $\gamma'=\tan.\,\phi'=\phi'$, &c. very nearly,

$$m\,\surd\,a.\phi\,\delta\,\phi+m'\,\surd\,a'.\phi'\,\delta\,\phi'+m''\,\surd a''.\phi''\,\delta\,\phi''+\&c.=0,$$

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* The two first terms in the value of F are excluded in the computation of $\frac{d\,F}{d\,\theta}$, $\frac{dF}{d\phi}$, because they are not functions of θ and ϕ.

and integrating,

$$^{*}\,m\sqrt{a}\,.\,\phi^2 + m'\sqrt{a'}\,.\,\phi'^2 + m''\sqrt{a''}\,.\,\phi''^2 + \text{\&c.} = K,$$

which is a theorem similar to the one of p. 435, and from which like inferences may be drawn.

For, if we take from the Tables of the inclinations of the orbits of planets (see *Astron.* p. 286.) the values of ϕ, ϕ', ϕ'', &c. such as they were at the epoch of 1800, and thence compute, as in p. 435, the value of K, it will be found to be a very small quantity. Now such value is the maximum and limit of the sum of $m\sqrt{a}\,.\,\phi^2$, $m'\sqrt{a'}\,.\,\phi'^2$, &c. consequently, ϕ, ϕ', must always be very small quantitities, since each of the preceding products can never, in the most extreme case, exceed K.

The system of the planets, then, with regard to the planes of their orbits, is perfectly stable. The planes are not fixed, indeed, but oscillate about a mean state of inclination. The limits of their oscillations are defined by this theorem: namely, that the sum of the squares of the inclinations multiplied respectively by the masses of the planets and by the square roots of the major axes is invariably the same.

On these properties, then, of the mean distances, the eccentricities and the inclinations, the *stability* of the planetary system depends. It is not necessary to it that the perihelia and nodes should either be stationary, or oscillate about their mean places, or move uniformly.

It must be recollected that the preceding results relate to the

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[*] If sin. ϕ which is nearly equal to γ, had been taken to represent it, then, since $\int \sin.\,\phi\,\delta\phi = -\cos.\,\phi$ the theorem would have been of this form,

$$m\sqrt{a}\cos.\,\phi + m'\sqrt{a'}\cos.\,\phi' + \text{\&c.} = k,$$

$$\text{or}\quad m\sqrt{a}\,.\,\sin^2.\,\frac{\phi}{2} + m'\sqrt{a'}\,.\,\sin^2.\,\frac{\phi'}{2} + \text{\&c.}$$

$$= \frac{1}{2}(m\sqrt{a} + m'\sqrt{a'} + \text{\&c.}) - \frac{k}{2} = k'$$

from which, inferences similar to those in the text, may be drawn.

secular variations of the elements. The mean distance, as it is plain, from the formula of p. 457, although exempt from a secular, is subjected to a periodical inequality: which, depending on the *configuration* of the disturbed and disturbing bodies, augments, to a certain extent, the mean distance, and then, by like degrees of diminution, causes it to return to its former magnitude.

The eccentricities, perihelia, inclinations and nodes also, besides their secular, are subject to periodical inequalities, which may be computed from the expressions of pp. 404, 416.

The periodical inequalities of the longitude, latitude and parallax of a planet, as well as the inequalities, periodical and secular, of the elements of its orbit, are produced by the disturbing forces of the other planets. Those disturbing forces are, in fact, but under peculiar circumstances of action, their attractive forces. These latter, at a certain distance; are, according to Newton, proportional to the masses of the attracting or disturbing bodies. Contrariwise, the perturbations expound the masses, and, in the analytical expressions of the three kinds of perturbations above specified, the mass of the disturbing body must enter as an indeterminate quantity. If therefore the quantity of perturbation, periodical or secular, be given by observation, the means are thence afforded of determining the mass.

According to mere theory it is indifferent which is the inequality we select for determining the mass. But in practice we are restricted to two: the periodical inequality of the disturbed planet's place, and the *secular* inequality of an element of its orbit.

We may determine the mass of Venus from the inequality produced by it, in certain situations, in the Earth's longitude, or from the secular inequality of the longitude of the Earth's perihelion. Both these can be determined by observation: the latter possesses magnitude because it is an accumulated effect. But the periodical inequality of the perihelion is a quantity far too minute for observation. It must be viewed merely as a theoretical result.

The practical method, however, of determining the mass of a disturbing planet is not quite so simple as we have stated it.

Jupiter and Mars, as well as Venus, interfere in disturbing the elliptical quantity of the Earth's longitude, and the place of its nearest distance from the Sun. If we consider the masses of these bodies as three indeterminate quantities, we must, in order to determine them, use three observations at the least. We may use more: indeed, it is plain, that, the greater the number of observations, (supposing them to be equally accurate) the more exact will be the determination of the masses.

It is of no consequence, in the method which has been described, whether the planet, the mass of which is to be determined, be with or without a satellite. But the mass of a planet of the first kind may be determined most simply (see p. 29.) from the greatest elongation and period of its satellite. There are then, at the least, two methods for determining the mass of Jupiter*: we may, therefore, use the two methods, the one to serve as a check on the other; or, in determining the mass of Venus from some inequality either in the Earth's motion, or in an element of its orbit, we may contract the investigation by assuming the mass of Jupiter to be that which is determined by means of the period and the greatest elongation of one of his satellites.

The mass of a planet that has no satellite must be determined from the effect of its disturbing force; the mass of a planet accompanied by a satellite may be determined by the effect either of its disturbing, or of its attracting force. But, in each case, the principle of the determination is precisely the same. That by which we measure the mass of a planet or the number of its particles, is some effect of their attraction. In one case the effect is the deflection of a satellite from its rectilinear course, and that effect is denominated *attraction:* in the other case, the effect is the deflection of another planet from its elliptical course; or, from that course which it would pursue did it obey solely the laws of projection and of its centripetal force: and this effect is denominated *perturbation*. The particles, in the two cases, exert their attraction under different circumstances, and the respective effects of their attractions are conveniently distinguished by different denominations.

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* Its mass may be determined by comparing, with the best observations, the *great inequalities* (see Chap. XIX.) which its action produces in the motion of Saturn.

In the following Chapter we will enter more into the details of the methods by which the masses of the Earth, the Moon, the Planets and the Satellites are determined.

CHAP. XXIV.

On the Method of determining the Masses of Planets that are accompanied by Satellites. Numerical values of the Masses of Jupiter, Saturn, and the Georgium Sidus. The Earth's Mass determined. The Methods for determining the Masses of Venus, Mars, &c. and, generally, of Planets that are without Satellites. The Masses of Satellites and of the Moon determined.

THE principle of determining the mass of an heavenly body, whatever it be, Sun, Moon, or Planet, is, as it has been already stated in the close of the preceding Chapter, precisely the same. Under different denominations, because under different circumstances, it is, in every case, some effect of the attracting particles of matter which serves to expound their number, or the mass of the body which they are supposed to constitute. The effect, however, as it has been already stated, in one class of instances, is *centripetal force:* in another, a *force that disturbs:* in the former, the effect, according to certain preconceived notions, is regularity, or, the equable description of areas and the observance of Kepler's Laws: in the latter, irregularity, or the perturbation of areas, the *progression* of the apsides, &c. And such a distinction in the effects of gravitation naturally suggests a convenient distribution of the methods of finding the masses of the heavenly bodies into two classes; one appropriated to the Sun and those Planets that have satellites: the other to Mercury, Venus, Mars, the Moon, and the satellites of Jupiter and Saturn.

To begin with the methods of the first class. These methods are contained in the formula of p. 29; according to which

$$P = \frac{360^0}{\sqrt{\mu}} \times a^{\frac{3}{2}}$$

μ denoting the attraction residing in, or transferred (see pp. 42, 43, &c.) to the central body, P the period, and a the mean distance of the revolving body.

μ the attraction of the central body, (if the revolving body be supposed a material point, or if its mass, relatively to that of the central body, be supposed insignificant) is proportional to its mass. It is, in fact, (see pp. 42, 43.) proportional to the sum of the masses of the central and revolving bodies. Let 1 denote the Sun's mass, M, Jupiter's, m the mass of Jupiter's fourth satellite; and, moreover, let A, a, P, p, denote, respectively, the mean distances and periods of Jupiter and his satellite: then, by the preceding formula,

$$1 + M = \frac{A^3}{P^2} \times 360^\circ,$$

$$M + m = \frac{a^3}{p^2} \times 360^\circ,$$

$$\text{consequently, } \frac{M + m}{1 + M} = \frac{a^3}{A^3} \times \frac{P^2}{p^2},$$

or,

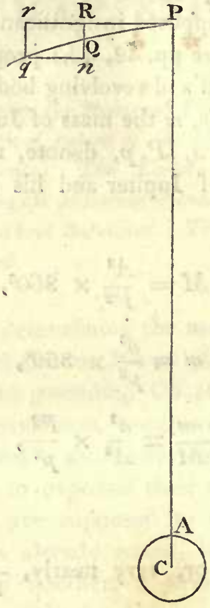
$$\frac{M}{1 + M}\left(1 + \frac{m}{M}\right), \text{ or, very nearly, } \frac{M}{1 + M} = \frac{a^3}{A^3} \times \frac{P^2}{p^2};$$

$$\text{whence } M = \frac{\frac{a^3}{A^3} \cdot \frac{P^2}{p^2}}{1 - \frac{a^3}{A^3} \cdot \frac{P^2}{p^2}},$$

from which formula M the mass of Jupiter, or the relative quantity of his matter compared with that of the Sun's, may be computed. And, as it is plain, the same formula will serve for determining the masses of Saturn and of the Georgium Sidus.

The preceding formula expresses the mathematical dependence of the mass of the attracting on the period of the revolving body. But the process which establishes that formula does not render im mediately obvious their necessary dependence. That, however, may easily be thus shewn.

Let *C* be the centre of the attracting body, *P* the revolving body; *PQ* a portion of its orbit, and *PR* a tangent to the orbit at the point *P*. Now, according to theory, *PQ* is described by virtue of the projectile motion *PR* and *RQ* the centripetal force: which latter arises from the attraction of the mass at *C*, and, at a given distance, is proportional to that mass. Suppose the mass to be increased, then *RQ* would be increased: it might become, in the same time, Rn ($= rq$). The orbit, therefore, could not remain circular (supposing for simplicity of illustration that to be its form) except the arc *PQ* became *Pq*. All portions of the orbit similar to *PQ*, would, in like manner, be increased by the increase of the mass at *C*: consequently, the number of portions of the arc described in the same number of portions of time would be diminished: the period, therefore, which is formed of such portions of time, would itself be less.



In the same way it would follow that the orbit, supposing it to retain its form, would necessarily be described in an increased period by a diminution of the mass of the attracting body.

But to return to the formula of computation: that may be, conveniently, thus modified: let s be the sine of the angle under which, at the planet's mean distance from the Sun, the mean radius of the satellite's orbit is seen: then $s = \frac{a}{A}$, and, consequently,

$$M = \frac{s^3 \frac{P^2}{p^2}}{1 - s^3 . \frac{P^2}{p^2}}$$

$$= s^3 \frac{P^2}{p^2} + \left(s^3 \frac{P^2}{p^2} \right)^2 \text{, very nearly,}$$

Suppose it were required to find Jupiter's mass from this expression: and by means of the elongation of his fourth satellite: then

$s = \sin. 8' \ 15''.85$ .... log. = 7.3809246

3

log. s^3 = 22.1427738

$P = 4332^d.6022$....log. 3.6367488

$p =$ 16.6888....log. 1.2224251

2.4143236

2

log. $\frac{P^2}{p^2}$ = 4.8286472

log. s^3 = 22.1427738

log. $s^3 . \frac{P^2}{p^2}$ 26.9714210; ∴ No. = .00093631

2

log. $\left(s^3 . \frac{P^2}{p^2} \right)^2$ 53.9428422......No. = .00000087

.00093718

therefore Jupiter's mass = .00093718, or $= \frac{1}{1067 \cdot 03}$,

the Sun's mass being 1.

Modern observations have added nothing, since Newton's time, to the accurate determination of Jupiter's mass.

Newton, from an elongation ($= 8' \; 16''$) of the fourth satellite determined by Pound, found Jupiter's mass equal to $\frac{1}{1067}$ *; a

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* Newton, and his very learned commentators, Le Seur and Jacquier, in determining the relative masses of Jupiter and the Sun, do not use as a kind of mean term, Jupiter's period but Venus's. This, without any gain of accuracy, occasions, in the process of computation, an additional step: for let $\mathfrak{Q}$ and d be the period, and mean distance of Venus, then,

$$\frac{\mathfrak{Q}^2}{P^2} = \frac{d^3}{A^3}, \text{ and } s^3.\frac{P^2}{p^2} = s^3\frac{\mathfrak{Q}^2}{p^2}.\frac{A^3}{d^3},$$

the factor $\frac{A^3}{d^3}$ causes (see p. 469.) the additional step, and $\frac{\mathfrak{Q}^2}{p^2}$ is not better known than $\frac{P^2}{p^2}$. But, if we look to the grounds of the methods, their principle is precisely the same: and we should in vain seek elsewhere for a more simple and clear illustration of Newton's Theory: of, at once, the principle and the Law of Gravitation. By the first, the masses of the Sun and Jupiter are to each other respectively, as the descents, from equal distances, in equal times, of two material points, or corpuscules, towards those bodies. At unequal distances, the descent from the less distance, must, by the second part of Newton's Theory, or the Law of Gravity, be diminished, in the ratio of the square of the greater to the square of the less distance, in order relatively to expound Jupiter's mass. But, in point of fact, there are no single corpuscules that separately descend in right lines towards the Sun and Jupiter. In order, therefore, to reduce the preceding principles to computations, there are requisite two preliminary conditions. The first consists in assuming, by reason of their relative minuteness, Venus (or any other planet) and Jupiter's satellite as the two corpuscles or material points placed at the distances of Venus from the Sun, and of the satellite from Jupiter. The second consists in assuming the deflections of Venus ($\mathfrak{Q}R$, $q\,r$) and the satellite from the tangents of their orbits for the rectilinear descents; and then we have

$$\text{Sun's mass} : \text{♃'s mass},$$

$$:: QR : qr \times \left(\frac{\text{rad. of orbit of ♃'s satellite}}{\text{rad. ♀'s orbit}}\right)^2.$$

result nearly the same as the preceding. There must here, therefore, be either coincidence, by chance, between modern and antient observations, or, in respect of finding an elongation, the former have no advantage above the latter.

There is not, however, an equally near agreement between Saturn's mass as it is now determined, and as it was determined by Newton; for, if we take 2′ 59″ to be the greatest elongation of the sixth satellite, we have

Logarithms.

P = 10758^d.96984 ... 4.0317707
p = 15.9453.... 1.2026327
2.8291380
2
5.6582760
3 log. sin. 2′ 59″ 20.8152834
26.4735594....No. = .00029755
2
6 log. sin. 2′ 59″ .... 52.9471188....No. = .0000000885
.000297638

therefore Saturn's mass = .00029764, nearly, or, $\frac{1}{3359.4}$.

Instead of 2′ 59″, Newton assumes the value of the greatest elongation to be 3′ 4″, and thence determines Saturn's mass to be equal $\frac{1}{3021}$ *. The difference arises principally from the difference in the assumed values of the greatest elongations. But here, for more than one reason, the modern observation is to be relied on.

The question, however, concerning the mass of Saturn does

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* The difference in these values is, according to one mode of considering it, enormous, being about ten times the mass of the Earth.

not solely rest on one kind of observation; and the uncertainty arising from a want of precision in the determination of the greatest elongations of his satellites may be, in degree at least, removed by an examination of the *great inequalities* which Saturn causes in Jupiter. These inequalities can be observed and computed: but not computed except by assuming a certain value to represent Saturn's mass: which assumed value, may, therefore, be corrected by comparing the results of observation and theory.

The theory of Jupiter and Saturn presents us also with other like expedients for correcting the assumed or approximate values of their masses. For, as we have seen (see Chap. XXII.) their secular inequalities arise principally from their mutual action.

In order to determine the mass of the Georgium Sidus, we have, the greatest elongation of his fourth satellite equal to $44''.2$, $P = 30688{\cdot}712687$, $p = 13.4559$

$$\begin{array}{rl}
\log.\ 30688{\cdot}7 = & 4.4869786\\
\log.\ 13.4559 = & 1.1289128\\
\cline{2-2}
 & 3.3580658\\
 & 2\\
\cline{2-2}
 & 6.7161316\\
3\ \log.\ \sin.\ 44''.2 \ldots & 18.9929385\\
\cline{2-2}
 & 25.7090701 = \log.\ .000051177\,;
\end{array}$$

therefore the mass of the Georgium Sidus equals, nearly,

$$.000051177, \text{ or } \frac{1}{19540}.$$

The mass of the Earth, since it is accompanied by a satellite, may be determined by the preceding formula. But, in one part, the process may be rendered more simple: or, we may go back to the very principle of the method, by expounding the Earth's mass, or its attraction, not by a merely computed space, (the sagitta of an arc described), but by a space actually passed through. The descent * of a heavy body at the Earth's surface diminished in the ratio of the square of the radius of the Earth's orbit to the square of the Earth's radius, and the *computed* descent of a heavy body from the Earth towards the Sun (which computed descent is the deflection of the Earth from the tangent of her orbit) will expound, very nearly, the relative masses of the Sun and Earth. They will not expound those masses *exactly* because the latter space is due not solely to the Sun, but to the joint attractions of the Sun and Earth: let s denote the latter space or deflection, in one second of time, of the Earth from the tangent of her orbit, and let g denote the space descended through towards the Earth, at the Sun's mean distance, then the Sun's mass is more truly expounded by $s - g$,

$$\text{and } \frac{\oplus\text{'s mass}}{\odot\text{'s mass}} = \frac{g}{s-g} = g \times \frac{1}{1-\frac{g}{s}}.$$

Now, $16\frac{1}{8}$ feet is the descent of a heavy body at the Earth's surface, in one second of time in a latitude the sine of which $= \frac{1}{\sqrt{3}}$: the number of feet in the Earth's radius is 20929656; therefore, $\frac{16\frac{1}{8}}{20929656}$ is that fractional part of the Earth's radius, which, at the Earth's surface, expounds the Earth's attraction: and

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* The actual space passed over by a falling body in one second of time, if it could be exactly noted, would not strictly expound the Earth's attraction; because, if the observation should be made at any place not at the pole of the Earth, the centrifugal force of rotation interfering with gravity would diminish its effect. We must, therefore, according to the latitude of the place of observation, compute the centrifugal force, its effect in the direction of gravity, and add that effect to the descent of a body in 1″ computed from the length of a pendulum. The sum will expound the force of gravity at the place of observation. It is usual to consider the place to be situated in that parallel the sine of which is $\frac{1}{\sqrt{3}}$; because the radius drawn thence to the centre of gravity of the terrestrial spheroid, is the radius of a sphere which, with the Earth's mean density, is equal to the Earth's mass.

$$\frac{16\frac{1}{8}}{20929656} \times \frac{\text{rad. of } \oplus}{\text{rad. of } \oplus\text{'s orbit}},$$

is the same space, but expressed by a fractional part of the radius of the Earth's orbit, which expounds the same attraction; but see p. 473,

$$g = \frac{16\frac{1}{8}}{20929656} \times \frac{\text{rad. of } \oplus}{\text{rad. of } \oplus\text{'s orbit}} \times \frac{(\text{rad. of } \oplus)^2}{(\text{rad. of } \oplus\text{'s orbit})^2}$$

$$= \frac{16.125}{20929656} \left(\frac{\text{rad. of } \oplus}{\text{rad. of } \oplus\text{'s orbit}}\right)^3$$

$$= (\text{see } \textit{Astron.} \text{ p. } 102.)\ \frac{16.125}{20929656} \times \sin^3.\ 8''.8 = \frac{5.9634}{10^{20}}.$$

In order to find the value of s, we have the arc (z) described in one second of time equal to

$$\frac{2 \times 3.14159}{365^d.256384 \times 24 \times 60 \times 60},$$

the mean radius of the Earth's orbit, being 1;

$$\text{consequently, } s = \frac{z^2}{2} = {}^{*}\frac{1982016}{10^{20}};$$

$$\therefore s - g = \frac{1982010}{10^{20}},$$

$$\text{and } \frac{\oplus\text{'s mass}}{\odot\text{'s mass}} = \frac{5.9634}{1982010}.$$

By the above methods, (all which are in fact the same) the masses of planets, that have satellites, are determined. They rest, as it has been seen (see p. 466.) on Kepler's Laws. But the methods by which the masses of planets unaccompanied by satellites, and of satellites themselves, are found, are in one respect so far unlike the preceding, that they depend on *digressions* from those Laws * ; for the Laws are strictly true only in a system of two bodies: and every inequality arising from the attraction of a *third* body is a species of perturbation, and causes digressions from those Laws. But although, after the establishment of a conventional language, we are thus enabled to distinguish and characterize the methods, yet their principle, which is Newton's Principle of Gravitation, is precisely the same. Some space, the increment of an angle or of a radius vector, either a momentary or an accumulated effect, expounds, in every case, a planet's attraction, whether it be regarded as central or external. A particle of matter at L describing round T as a central body the arc LQ, QR the deflection from the right-lined course LR, expounds the attraction and mass of T. If in the time of describing LQ, a particle at L would descend towards S through a space equal to Ln, Ln would expound, at the distance LS, the mass of S; and $Ln \times \left(\frac{LS}{LT}\right)^2$ and QR would expound the relative masses of S and T. The above are momentary effects. Now we cannot conveniently compute such effects of disturbing masses. We are obliged then, to have recourse to their accumulated effects: either to the periodical inequalities with which one planet affects the radius vector and longitude of another, or to the secular inequalities with which it affects its elements.

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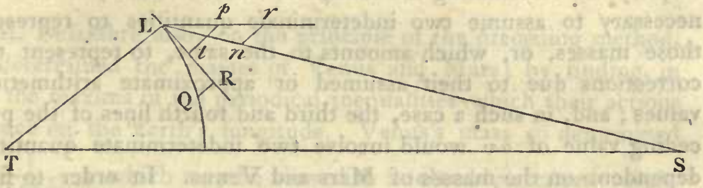
* Computation of z^2

log. 3.14159 = 4971500.................. 4971500
———— 7.4991114
————
log. 365.256384 ...... 2.5625977......... 2.9980386
log. 3600................3.5563025 2
log. 24....................1.3802112 ————
———— 5.9960772 (20 borrowed)
7.4991114; .... ∴ No. = $\frac{991008}{10^{20}}$.



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* If a person were in search of paradoxes he might find one here, and state that one method was founded on the truth, the other on the falsehood, of Kepler's Laws.

The Earth's place, for instance, is sometimes behind and at other times before its elliptical place, by the action of the planets. The deviation from its elliptical place (see p. 311.) is

$$\begin{aligned} \delta v = {} & 8''.9 \sin.(\text{☽} - \odot) \\ & + 7''.059 \sin.(\text{♃} - \odot) - 2''.51 . \sin. 2(\text{♃} - \odot) \\ & + 5''.29 \sin.(\text{♀} - \odot) - 6''.1 . \sin. 2.(\text{♀} - \odot) \\ & + 0''.4 \sin.(\text{♂} - \odot) + 3''.5 . \sin. 2.(\text{♂} - \odot), \end{aligned}$$

in the deduction of which formula, certain values were assumed for the masses of the Moon, Jupiter, Mars and Venus. But, in an enquiry for determining the masses of the two latter, it will be necessary to assume two indeterminate quantities to represent those masses, or, which amounts to the same, to represent the corrections due to their assumed or approximate arithmetical values; and, in such a case, the third and fourth lines of the preceding value of δv would involve two indeterminate quantities dependent on the masses of Mars and Venus. In order to find two such quantities, we must, at least, form a second equation (for the value of δv) similar to the preceding.

The comparison then of the values of δv computed and found by observation will furnish two equations for determining the masses of Venus and Mars. But in an enquiry so delicate where the error of a second so materially affects the result, it will be expedient * to compute, from theory and observation, many values of δv, and to determine the masses from the means of several results.

Clairaut, in that excellent Memoir (*Mem. Acad. Paris*, 1754.) which has been more than once alluded to, uses the method which has been just described, founded on the periodical inequalities of the Earth, for determining the mass of Venus. He supposes that the actions of Mars and Mercury in altering the Earth's place may be neglected: and he investigates those positions of the Sun and Moon in which the perturbation arising from the latter body are nothing. In such positions, then, the difference between the Earth's place observed and computed from previously established conditions, would expound the attraction of Venus and serve to determine her mass. It would at least serve (and this is almost all that it will do) to furnish one out of many results by which the mass of Venus is to be determined. There is in these enquiries, as we have already stated, matter of great uncertainty, and Clairaut regards the result of his method merely as an essay * towards the determination of the mass.

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* 'Dans cette rencontre comme dans beaucoup d'autres telles que la fixation du lieu moyen, de celui des apsides, &c. le nombre des observations peut bien reparer l'incertitude qui est dans chacune d'elles.' *Mem. Acad.* 1754, p. 523.

M. Delambre also, on the principle of the preceding method, has determined the masses of Venus and Mars; by finding, in fact, the maxima of the periodical inequalities which their actions produce on the Earth's longitude. Venus's mass so determined is $\frac{1}{356632}$, which may be viewed as a tolerably accurate result, since the periodical inequalities, from which it is derived, were determined by a great number of good observations. But the secular inequalities, were they exactly known from observation, would best serve for determining the masses of planets that have not satellites. The diminution, for instance, of the obliquity of the ecliptic and the progression of the Earth's perihelion, are secular inequalities, and arise from the disturbing forces of the planets, principally, from those of Venus and Jupiter: if they solely so arose, then, since the mass of Jupiter (see p. 469.) is, by other means, known, one or more observations of the diminution of the obliquity of the ecliptic, or of the progression of the Earth's perihelion, compared with the computed results of those inequalities from an assumed value of Venus's mass, would serve to determine the error of such assumed value, and, therefore, in

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* 'La masse de Venus est environ deux tiers de celle de la Terre. On sent bien que cette determination ne peut etre regardée que comme un essai: il faudroit faire un comparaison plus ample de la Theorie avec les observations, pour pouvoir etre entirement satisfait sur une matière aussi delicate.' *Mem. Paris*, 1754, p. 561.

fact, the mass of Venus. And the method is the same, only longer, if, as is really the case, we consider the diminution of the obliquity of the ecliptic to be produced by the action of all the planets. In such a case, we must, in *computing* the diminution, assume three indeterminate quantities, to represent, respectively, the masses of Mercury, Venus, and Mars, the three planets that are unaccompanied with satellites. Then the results, at least three in number, compared with as many observations, would serve to determine the quantities that represent the masses.

If we recur to p. 449. we may easily illustrate this method.

The mean annual diminution of the ecliptic is there expressed by

$$-(0,1)\tan.\phi'\sin.\theta'-(0,2)\tan.\phi''\sin.\theta''-(0,3)\tan.\phi'''\sin.\theta'''+\&c.$$

and in decimals of seconds by

$$-.008521-.30995-.010336-.158234-.013821.$$

Now, in this computation, the only quantities not known with sufficient accuracy either by direct observation, or by deduction from observation, are the masses of the planets: and these are necessary in computing (0, 1), (0, 2), &c. for (see p. 445.),

$$(0,1)=m'.\frac{n\,a^2}{4\,a'^2}B'a'^3,$$

in which n, a, a' are known by observation and Kepler's Law, (see p. 29.) and B' may be computed by the methods of Chapter XVIII: ϕ', θ', &c. are known by observation. The numerical value .008521, then, was deduced by *assuming*, as an approximate value, $m'=\frac{1}{2023810}$, and the next number by assuming $m''=\frac{1}{383157}$ as the approximate value of the mass of Venus; and the third number (.010336) was deduced by assuming $m'''=\frac{1}{1846082}$ as the approximate value of the mass of Mars. The fractions representing the masses of Jupiter and Saturn, and used in the preceding process, were deduced by the methods of p. 469. They may be considered to be more accurately determined than m', m'', m'''; but suppose all the values of the masses, the assumed and deduced, to stand in need of correction: and, instead of the preceding values of m', m'', &c. let

$$\frac{1}{2023810}+\frac{\mu'}{2023810},\ \frac{1}{383157}+\frac{\mu''}{383157},\ \&c.$$

represent them; then, the mean annual diminution of the obliquity of the ecliptic (δE)* will be

$$-0''.500862$$

$$-.008521\mu'-.30995\mu''-.010336\mu'''-.158234\mu^{\text{iv}}$$

$$-.013821\mu^{\text{v}},$$

and if we consider the masses of Jupiter and Saturn to be accurately determined, μ^{iv}, μ^{v}, each $=0$, and the preceding equation will then contain only three indeterminate quantities μ', μ'', μ'''.

In order to determine these quantities, three equations, at the least, are requisite: we must, then, compute for two other epochs (the epoch of 1750 is the one belonging to the above computation) two other formulæ similar to the preceding, in which, since ϕ', ϕ'', &c. θ', θ'', &c. would have values different from the former ones (see p. 450.): the numerical coefficients of μ', μ'', μ''' would be different, and their sums, the *computed* diminutions, would be different from .500862. the computed diminution for 1750. This operation would then give us three computed values for δE (see p. 448.) which compared with three values, for the respective epochs, deduced from observations, would furnish three equations for determining the three corrections μ', μ'', μ'''.

If the masses of Jupiter and Saturn be considered as not sufficiently correct, we must deduce, at the least, five computed and five observed values of δE. The determinations of μ', μ'', &c. cannot be effected by a less number of equations: but it is plain a greater number (10, 15, 20, &c.), provided the observations were equally good, would lead us to more certain results.

But the fact is, there does not exist a sufficient number of good observations for the exact settling of this point. The observations required are of the nicest kind, since the question is concerning the fractions of a second. None but the best instruments have any concern with its determination; inferior instruments serve only to perplex it: and, if we needed a sort of practical proof of the incertitude that still remains on this subject, it would be sufficient to state that* M. Delambre, in his late Treatise, states the secular diminution of the obliquity of the ecliptic to be 50″, whereas Mr. Pond, by means of the new Mural Circle (see *Nautical Almanack* for 1818) makes it 40″.

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* More correctly $\delta\Delta E$.

The mass of Venus, then, which principally causes the diminution of the obliquity of the ecliptic, cannot, thence, from a defect of existing observations, be very accurately determined. We must wait for future observations; in the mean time, Delambre, as we have said, is of opinion that the masses of Venus and Mars may be best determined from the periodical irregularities they produce in the Earth's motion.

The mass of Mercury, (which indeed has little influence either on the periodical or secular inequalities of the planets) is by far more uncertain than that of Venus. It is determined altogether on conjectural grounds and by analogy. The densities of the other planets, it is found, are, nearly, inversely as their mean distances from the Sun. If Mercury's density be *assumed* according to this observed law, then from astronomical measurements of his diameter we may determine his mass. The result, however, is an uncertain one, but, luckily, nothing that is important in Astronomy depends upon it.

The diminution of the obliquity of the ecliptic, is, in other words, (see p. 446.) a change in the inclination of the plane of the Earth's orbit produced by the action of the other planets. The inclination of the plane of any planet, and, consequently, the obliquity of its ecliptic, is, in like manner, changed by the disturbing force of the Earth and the other planets. Its variations, then, must expound, like the preceding, (see p. 479. l. 4, 5, &c.) the masses of disturbing planets. And, viewed mathematically, they are equally sufficient, but, practically, more unfit than the preceding secular inequalities, to determine those masses. This is one of the points on which it is necessary that theory should have (as it may be said) a *communication* with observation, in order to prevent its being fruitlessly embarrassed in the investigation of useless minutiæ.

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* 'Quelques Astronomes ont voulu pendant un tems nier toute diminution: forcés d'en adopter une, ils la faisaient beaucoup moindre. Lalande, apres l'avoir fait beaucoup plus forte, a cru long-tems qu'elle n'etait que de 33″: il a fini par supposer 50″, et nous ne sommes guères plus avancés aujourd'hui.' Tom. III. Chap. xxxii. Art. 11.

The secular inequalities of the perihelia and nodes stand, with regard to their theoretical significancy, in the same predicament as the secular diminution of the obliquity of the ecliptic, and the secular variations of the inclinations of the planes of orbits. They arise from attraction, and may serve to expound the attracting masses: in fact, equations exactly similar to the preceding, expound the progressions of the perihelia, when the assumed masses, for the purpose of deducing the corrections, are multiplied, respectively, by $1 + \mu'$, $1 + \mu''$, $1 + \mu'''$, &c. The progression of the Earth's perihelion, for instance,

$$\text{or, } \frac{\delta \pi}{dt} = 11''.949 \, *$$

$$- 0''.4149 \, \mu' + 3''.8135 \, \mu'' + 1''.5461 \, \mu''' + 6''.804 \, \mu^{\text{iv}}$$

$$+ .19406 \, \mu^{\text{v}} + .0064 \, \mu^{\text{vi}},$$

μ^{vi}, being the correction for the mass of the Georgium Sidus.

This inequality may be used for correcting the mass of Venus.

The masses of Jupiter's satellites must be determined on the same principle that the masses of planets, without satellites, are: that is, either by the periodical or the secular inequalities that arise from their mutual perturbations. An inequality of the first kind, the *variation* of the first satellite produced by the action of the second serves to determine the mass of the second. The *variation* (see p. 363.) is thus expounded,

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* We must bear in mind the remark of p. 479. When Halley succeeded Flamstead in the Royal Observatory, he found not a sufficient number of good observations to enable him to determine the *species* and *position* of the Earth's orbit: 'much less, say the Authors of the Preface to Halley's Tables, could he discover an equation to the motion of its aphelion, or the other small equations by which its orbit is affected: for these are not to be found out, nor their quantities determined, but by a long series of the nicest observations.'

$$\delta v = \frac{m' n F}{n - 2n'} \sin. 2 (n' t - nt + \epsilon' - \epsilon).$$

F being computed, and δv, n, n', &c. being known from observation, m' may be determined. In its general character, the case is like that in which it is proposed to determine the mass of Venus from the inequality it produces in the Earth's longitude: but, in one respect, a difference is to be noted between the two cases. The *variation* of a satellite of Jupiter, which is indeed its principal inequality, cannot be observed, under the same circumstances and situations that the Earth's variation can. The laws of the motions of satellites must be deduced, almost entirely, from their eclipses and occultations. Now, an inequality (such as the variation just mentioned) would cause an eclipse, computed according to the circular, or Kepler's Elliptical Theory, to happen either sooner or later than such computed time. The acceleration or retardation, therefore, of an eclipse of the first satellite caused by the disturbing force of the second, would serve, as we have seen in other cases, to expound the mass of the latter. For instance, it is found by observation, that the greatest acceleration and retardation of the eclipse of the first satellite, is, in time, equal to

$$3^m\ 13^s.0799, \text{ or, } 0^d.00223471,$$

consequently, since the synodic period of the first satellite is $= 1^d.769861$, the above quantity is $\frac{.00223471}{1.769861} \times 360^0$, and equal to $0^0.45453$.

Now, if according to the methods in Chapter XVII, we compute F, and thence $\frac{n}{n - 2n'} . F$, we shall find the latter equal to $\frac{1^0\ 57'\ 22''.66}{10000}$, accordingly $\frac{m' n}{n - 2n'} F$, the coefficient (or greatest value of the equation), or, what it now expounds, the greatest retardation of an eclipse, equals to

$$m' \, (1^{\circ}\ 57'\ 22''.66).$$

(supposing m' to designate ten thousand times the mass of the second satellite).

Equate this with $0^{\circ}.454553$, and

$$m' = \frac{.454553}{1^{\circ}.956296} = .2333,$$

nearly, the mass of Jupiter being supposed equal 1.

The principal inequality of the second satellite which (see Chap. XX.) is its *variation*, arises, almost entirely, from the actions of the first and third satellite. The greatest term of this inequality, then, would be expounded by an equation such as

$$A m + B m''.$$

The above quantities (m, m'') cannot be determined except by the aid of a second equation that should also involve them. The annual and sidereal motion of the apside of the orbit of the fourth satellite (the *Perijove*, as Bailly calls it), if it arose solely from the actions of the three other satellites, would furnish such an equation, of the form

$$A' m + B' m' + C' m'',$$

but equivalent, since m' is supposed to be previously determined, to an equation involving only two indeterminate quantities m and m''. The fact, however, is that the *oblateness* (applatissement) of Jupiter has considerable influence on the motion of the *Perijove*. The same want or *defect of sphericity* influences also the motions of the nodes of the orbits of his satellites. In order then to determine this oblateness of Jupiter, we must employ a new equation; the annual motion, for instance, of the nodes of the second satellite. But the fourth satellite combines with the others, and with the *oblateness*, in producing this. One more equation, therefore, will still be necessary which involving m, m', m'', m''', and μ

(Jupiter's oblateness) and combined with the three other equations, will serve to determine

$$m', m'', m''', \mu.$$

The values of the masses of the satellites are

1st satellite	.0000173281,
2d	.0000232355,
3d	.0000884972,
4th	.0000426591,

and the ratio between the polar and equatoreal diameters of Jupiter (determined from the value of μ) is .9286992.

It is not a little remarkable that this ratio determined, on theoretical principles, agrees, almost exactly, with that (= .929) which is deduced from a mean of direct measurements of the least and greatest diameters of Jupiter.

The mass of the Moon, the Earth's satellite, cannot, it is evident, be determined as those of Jupiter's satellites have been. It requires a peculiar method, grounded, indeed, on the Principle of Gravitation, and on that modification of its action which is denominated Perturbation. And, of this kind, there are four principal effects produced by the Moon that present themselves as convenient means for measuring its mass: the tides; the *nutation* of the Earth's axis; the parallax of the Moon: *the Lunar Equation* (see p. 85.) of the Solar Tables *.

The first of these phenomena (the perturbation of the waters of the ocean) was originally used by the great Author of *Physical Astronomy* (see *Princ.* Lib. III. Prop. XXXVII.) to determine the relative masses of the Sun and Moon. Laplace makes use of the observed tides in the Port of Brest for the same end. He thence makes the mass of the Moon $= \frac{1}{58.6}$ (the Earth's being 1.)

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* This *equation* is the correction to the inequality in longitude of the Sun caused by the Moon's disturbing force: and is the subject of Table X. in the Solar Tables, inserted in vol. III. of Vince's *Astronomy.*

Newton, (see Cor. 4. Prop. XXXVII. Lib. 2.) makes it $\frac{1}{39.788}$. Laplace, however, thinks that local circumstances influence the Moon's action on the tides in the harbour of Brest, and cause her resulting relative mass to be too large. He examines then the three latter phenomena (see p. 484.) for the purpose of diminishing and correcting the value of $\frac{1}{58.6}$.

The *Nutation* (see *Astronomy*, Chap. XVI.) arises from the Moon's action. The larger the mass of the Moon the larger will be the coefficient of the nutation. That coefficient, computed on the supposition of the Moon's mass being $= \frac{1}{58.6}$, is $10''.05$: but according to Maskelyne's observations, (see Maskelyne's Tables, and *Astronomy*, pp. 164, &c.) it is nearly $= 9''.6$: and, if this be considered to be the true value, the corresponding value of the Moon's mass would $= \frac{1}{71}$.

The Moon's parallax furnishes the second means of correcting her mass. The horizontal parallax is the ratio of the Earth's radius (D) to the mean distance of the Moon (a). Now this ratio $\left(\frac{D}{a}\right)$ may be computed by comparing, the versed sine of an arc of the Moon's orbit with the descent of a heavy body at the Earth's surface. But in such a computation, the Moon's mass (see Preface,) is an ingredient. The resulting numerical value then of the parallax, depends, in part, on the value assumed for that mass. The comparison then of the computed parallax, with the parallax deduced from observation, must needs furnish the means of correcting the assumed value. Thus, the Moon's mass being assumed $= \frac{1}{58.6}$ the computed parallax (the constant part of the expression for it) is $57' \; 8''.08$. But, by the comparison of numerous observations, that constant part is found equal to $57' \; 12''.03$ which corresponds to a mass of the Moon $= \frac{1}{74.2}$.

The maximum value of the *Lunar equation* (the value of its coefficient) was, in p. 85, stated to be $8''.8$: and this was deduced on the supposition (see p. 73.) that the Moon's mass is $\frac{1}{58.6}$, and the Sun's horizontal parallax $8''.812$. But if we take the coefficient of the Lunar equation to be $7''.5$, as Delambre has by the comparison of a great number of equations determined it to be, and the Sun's horizontal parallax to be $8''.56$, (which value agrees with most of the results obtained from the last passage of Venus over the Sun, and with a result obtained by Laplace from the Lunar theory) the corresponding value of the Moon's mass will be $\frac{1}{69.2}$.

The Moon's mass, then, from these three last phenomena, is less than what it results from observations of the tides in the harbour of Brest. Laplace considers $\frac{1}{68.5}$ to be the most probable value of the Moon's mass, that of the Earth's being called 1. This value makes the Moon's action on the tides to the Sun's as 2.566 is to 1.

Two of the phenomena, which have been just adverted to, for the purpose of determining the quantity of matter in the Moon, have, with regard to their cause and the law of their variation, found no place in the present Treatise. The Treatise, therefore, on that account, may be thought imperfect. It must be recollected, however, that its principal scope is a solution, and that in an extended sense, of the Problem of the Three Bodies. The Nutation of the Earth's axis, and the Tides, belong to a problem of a different kind: and it would be a most violent extension, and a most arbitrary as well as useless generalization, to include them within the former.

The questions, however, of the tides, the nutation of the Earth's axis, the precession of the equinoxes, the figure of the Earth, the variation of gravity in different parts of the Earth, the influence of the spheroidical figure of the Earth on the Moon's motions, the influence, generally, of the spheroidical figures of primaries on the motions, and on the elements of the orbits of their secondaries, are highly interesting, and, beyond a doubt, entirely within the province of Physical Astronomy. But, as it has been already stated, they form a class apart. They are connected with the former investigations of the periodical and secular inequalities of the motions of the Moon and the planets, inasmuch as they both depend, for their explanation, on Newton's Principle and Law of Gravity: and distinct from them, seeing that they rest on different dynamical principles, and require different formulæ of solution. Guided by this natural line of distinction, the Author of the present Volume here concludes it, after having gone through most of the investigations of the former class: those of the latter may, at some future period, furnish him matter for farther speculations.

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